Wreath Products of Generalized Automata*

Jaan Penjam† Varmo Vene‡

Institute of Cybernetics Tartu University

Various notions of automata are ubiquitous in computer science. The concrete formulations differ in details, but have a common basic "core", that of states and state transitions. Categories have been used to capture different classes of automata into a uniform framework at least since seventies [1]. More recent approach, based on coalgebras [2], has been very successful on characterizing dynamic behaviors. In this work, we are interested on structural properties of a class of automata. In particular, we introduce a generalization of automata with "memory" (see eg. [4]) and their wreath products that can be used as a general (de)composition operation.

A generalized automaton can be represented as a structure $\mathfrak{A} = (\mathbf{A} \xrightarrow{A} \mathbf{Set}; \Box)$ where $\mathbf{A}$ is a small category, A is a functor from $\mathbf{A}$ to $\mathbf{Set}$ and $\Box$ is a (partial) feedback operation

$$\Box : \coprod_{a \in \mathrm{Obj}(\mathbf{A})} a^A \times \mathrm{Mor}(\mathbf{A}) \longrightarrow \mathrm{Mor}(\mathbf{A}),$$

satisfying the condition

$$x \Box (f \cdot g) = f^A(x) \Box g$$

for all $f \in \mathrm{Hom}_{\mathbf{A}}(a, a')$, $g \in \mathrm{Hom}_{\mathbf{A}}(a', a'')$ and $x \in a^A$ with a, a' and a'' in $\mathrm{Obj}(\mathbf{A})$. Intuitively, the category $\mathbf{A}$ represents the transition structure of an automaton, where every state (ie. object of $\mathbf{A}$) has a "memory" attached to it by the functor A. The feedback operation $\Box$ describes the dependence between the transition structure and memory values.

A wreath product of two automata $\mathfrak{A} = ((\mathbf{A} \xrightarrow{A} \mathbf{Set}); \Box)$ and $\mathfrak{B} = ((\mathbf{B} \xrightarrow{B} \mathbf{Set}); \boxdot)$ is defined as $\mathfrak{A} wr \mathfrak{B} = ((\mathbf{A} wr \mathbf{B} \xrightarrow{AwrB} \mathbf{Set}); \boxtimes)$. Here the product of categories $\mathbf{A}$ and $\mathbf{B}$ is defined as a category $\mathbf{C}$ with the set of objects $\mathrm{Obj}(\mathbf{C}) = \{(\alpha, b) \mid b \in \mathrm{Obj}(\mathbf{B}), \alpha : b^B \longrightarrow \mathrm{Obj}(\mathbf{A})\}$, where the map α indicates for any element $m \in b^B$ some object $\alpha(m)$ in $\mathrm{Obj}(\mathbf{A})$. For any objects (α, b) and (α', b') in $\mathbf{C}$ a morphism $(\alpha, b) \xrightarrow{(\Phi, f)} (\alpha', b')$ is the pair (Φ, f) with f in $\mathrm{Mor}_{\mathbf{B}}(b, b')$ and Φ a collection of morphisms in $\mathrm{Mor}(\mathbf{A})$, such that

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†Institute of Cybernetics at Tallinn Technical University, Akadeemia tee 21, 12618 Tallinn, Estonia; jaan@cs.ioc.ee

‡Dep. of Computer Science, Tartu University, Liivi 2, EE-50409 Tartu, Estonia; varmo@cs.ut.ee

$$\forall m \in b^B, \qquad \alpha(m) \xrightarrow{\Phi(m)} \alpha'(f^B(m)).$$

The wreath product of functors A and B is a mapping $\mathrm{Obj}(\mathbf{C}) \longrightarrow \mathrm{Obj}(\mathbf{Set})$ such that

$$(\alpha, b)^{AwrB} = \{(l, m) \mid m \in b^B, l \in [\alpha(m)]^A\}$$

For any morphism $(\Phi, f) \in \mathrm{Mor}_{\mathbf{C}}((\alpha, b), (\alpha', b'))$, define $(\Phi, f)^{AwrB}$ as the map (Φ^A, f^B) in $\mathrm{Mor}_{\mathbf{Set}}((\alpha, b)^{AwrB}, (\alpha', b')^{AwrB})$.

Finally, having any morphism $(\Phi, f) \in \mathrm{Mor}_{\mathbf{C}}((\alpha, b), (\alpha', b'))$ and any element $(l, m) \in [(\alpha, b)]^{AwrB}$, define the feedback operation $\boxtimes$ by

$$(l, m) \boxtimes (\Phi, f) = (l \Box \Phi(m), m \boxdot f).$$

We will show that the wreath product above is well-founded in the sense that the result is a generalized automaton as well. We will discuss what specializations of this general product give some intuitively simple (parallel, serial etc.) compositions of automata. These general conceptions are also illustrated by the examples of specialization them to semigroup action systems, attributed automata etc.

Postscriptum This research was started by Uno Kaljulaid (1941-1999) in the mid of nineties, but was left unfinished due to his sudden death. This presentation is based on the early research report [3] and some of his unpublished notes. It is a part of a larger project by his former colleagues and students to preserve his scientific legacy and make it publicly available.

References

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