Review of Abstract Stone Duality

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Abstract Stone Duality seeks to unify recursion theory with general topology (the topological semantics of programming languages having made a link in one direction), including a recursive theory of "sets" (as discrete spaces or inductive types), coinductive types (as compact Hausdorff spaces) and exact real analysis. We deliberately reverse the relationship between sets and topological spaces, seeking a theory of topology *directly* — sets are merely a special case of spaces.

One way of developing topology in a recursive way was suggested by the study of function spaces, locally compact spaces, continuous lattices and domain theory. Whilst there are uncountably many points altogether (for example in $\mathbb{R}$), continuous functions are determined by their values on a countable "basis" (such as $\mathbb{Q}$). The category of spaces and continuous functions can be presented entirely in terms of such bases, albeit at the cost of some coding. A recursive version of topology is thereby obtained by requiring the basic elements to be numbered and the structure and functions to act via programs on these numbers.

Another logical reading that can be made of the basis presentation is to identify the strength of the meta-theory that it requires. Since continuous functions are represented by relations, a basis must be an object of a regular category or an allegory. Stable disjoint coproducts and either free monoids (lists), or free semilattices (finite subsets) and transitive closure of relations, are needed to encode other structure. When the category of spaces is made as general as the theory of bases allows, it also provides stable effective quotients of equivalence relations. The natural meta-theory is therefore that of an **arithmetic universe**, introduced by Joyal to give a categorical proof of Gödel's incompleteness theorem.

If the arithmetic universe is actually a topos, the category of "spaces" so constructed is equivalent to that of locally compact locales over the topos. Conversely, the standard views of a space as a set of points "equipped" with a topology, or of a locale as a set of formal open sets, imply that the universe is a topos. We claim, therefore, that we have the appropriate notion of (locally compact) topological space generalised from a set- or topos-theoretic meta-theory to recursion.

Traditional topology and locale theory both rely on arbitrary unions of open subspaces. We want to replace these by recursively enumerable unions, but without grafting traditional recursion theory on to the front of the theory. One way to achieve this is by means of the theory of bases that we have just described, in the case where the arithmetic universe is the free one. In place of the *ad hoc* traditional definition of general recursion, the axioms are all presented as universal properties, minimalisation in particular being a consequence of epi-mono factorisation. Nevertheless, it remains computationally objectionable because many constructions are encoded via nested lists.

The Stone dual of the category of spaces is captured by an algebraic theory, which has traditionally been formulated in an infinitary way over sets. Abstract Stone duality avoids the *arbitrary* unions by presenting the theory instead over the spaces themselves, using the monad arising from the contravariant self-adjunction $\Sigma^{(-)} \dashv \Sigma^{(-)}$, where Σ is the Sierpiński space. This is known as the continuation monad and has been used elsewhere to study computational effects.

Underlying this monad is a simply typed λ-calculus that is *restricted* in that we may only form exponentials like $\Sigma^{X_1\times\cdots\times X_k}$ and not Y^X. The monadic hypothesis adds both terms and types to this calculus. If $\Gamma \vdash P : \Sigma^2 X$ satisfies the equation saying that its transpose $\Sigma^X \to \Sigma^\Gamma$ is a homomorphism then $\mathsf{focus}\, P$ is the term a for which $P = \lambda\phi.\phi a$ (forming such a term without the equation leads to computational effects). Certain ("Σ-split") subtypes of X are encoded by terms $E : \Sigma^X \to \Sigma^X$ called nuclei that satisfy another equation.

In order to make the abstract monadic calculus agree with topology, Σ must be a lattice with existential quantification $\exists_N : \Sigma^N \to \Sigma$ when N is any space that we regard as a "set", in particular $\mathbb{N}$. The idea that Σ classifies both open and closed subspaces is captured by the Phoa principle, $F\sigma = F\bot \vee \sigma \wedge F\top$.A further equation is needed to provide fixed points etc.

ASD is equivalent to the theory of bases because a basis for X is presentation of it as a Σ-split subtype of Σ^N. However, ASD makes the conceptual and computational advance of providing a λ-calculus as an alternative to lists, which may themselves be encoded as λ-terms using modal logic.

The "λ-topology" view throws new light on familiar notions in general topology itself, notably **compactness**, which it defines by means of a universal quantifier $\forall_K : \Sigma^K \to \Sigma$. Dually, a space N that has $\exists_N : \Sigma^N \to \Sigma$ is called **overt**, a concept which is present in the traditional theory but barely visible there, as excluded middle makes *all* spaces overt. Discrete or Hausdorff spaces have $(=_X)$ or $(\#_X) : X \times X \to \Sigma$.

Theorem In any model of ASD, the full subcategory of overt discrete spaces is an arithmetic universe, and the given model is recovered by the basis construction above.

None of the universal properties of a pretopos went *in* to the axioms of ASD — indeed its category of spaces does not admit pullbacks — so it is very remarkable that we can obtain arithmetic universes in this way.

Overt discrete and compact Hausdorff spaces have lattice-dual definitions in ASD, so they have many similar properties, and each offers conjectures for new results about the other. They are the denotations of inductive and coinductive types, and I intend to develop a calculus to exploit this; it would have free (inductive) algebras with coinductive arities, and cofree coalgebras with inductive arities.

Turning to the computational side, the subtype information may simply be erased, leaving a λ-calculus with lattice operations and $\exists$, $=$, $\#$ and recursion over $\mathbb{N}$. The λ-calculus strongly normalises, whilst the other connectives apart from disjunction and recursion amount to unification, which also normalises into what is essentially a PROLOG clause. We obtain a whole program when we restore the disjunction and recursion (except that these are not interpreted correctly by the standard sequential (Warren) abstract machine). Whilst normalising the λ-calculus eliminates the "administrative" redexes that arise from the continuation monad, it also proliferates copies of parts of the program, so a way is needed to opt for a functional translation. Recursion at higher types also needs to pass λ-terms, so the target is actually λ-PROLOG.

Abstract Stone Duality has been funded by EPSRC grant GR/S58522/01 since September 2003. During the three-year course of this project I hope to develop calculi for inductive and coinductive types, and for "exact topology", that translate via ASD into λ-PROLOG programs.

The papers may be obtained from

`www.cs.man.ac.uk/~pt/ASD`

My presentation in Tallinn will be an "executive summary" of the ASD project. It is directly relevant to APPSEM Theme I, and indirectly to B, E and F.