# Wreath Products of Generalized Automata

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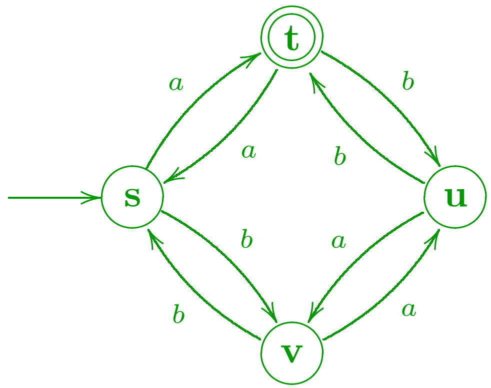
# FA as a Category

$$S = \{\mathbf{s}, \mathbf{t}, \mathbf{u}, \mathbf{v}\}$$
$$\Sigma = \{a, b\}$$




**Semigroup action:** $A = (S, \Sigma; \circ)$ where $a \circ (uv) = (a \circ u) \circ v$ holds for all $s \in S$ and $x, y \in \Sigma^\star$.

# FA as a Category

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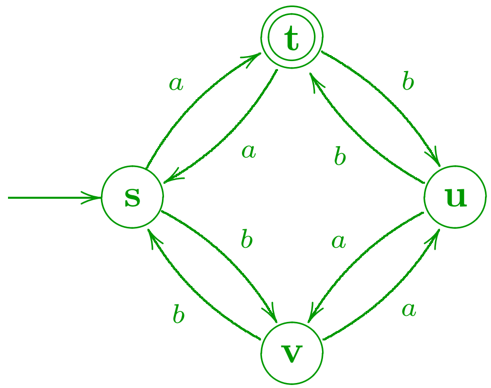
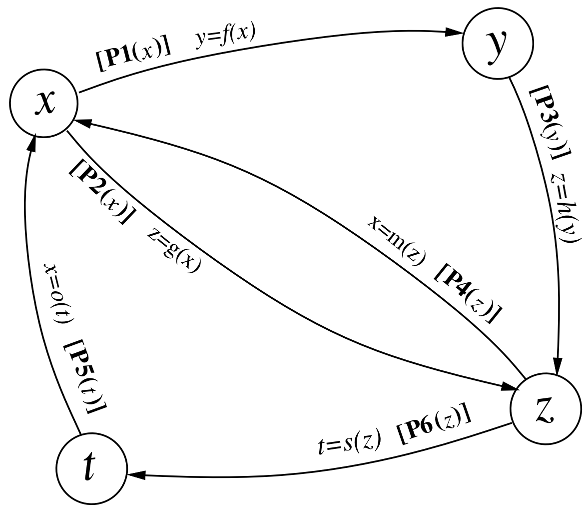



**Category Automaton:** $\mathbf{A} = (\mathrm{Obj}(\mathbf{A}), \mathrm{Mor}(\mathbf{A}))$ where

- $\mathrm{Obj}(\mathbf{A}) = S$
- $\mathrm{Mor}_{\mathbf{A}}(s, s) = \{s \xrightarrow{\varepsilon} s, s \xrightarrow{aa} s, s \xrightarrow{abab} s, \ldots\}$
- $\mathrm{Mor}_{\mathbf{A}}(s, t) = \{s \xrightarrow{a} t, s \xrightarrow{abb} t, s \xrightarrow{aaa} t, \ldots\}$
- $\ldots$

# Automaton with memory




$M = (S, T)$, where :

- $S$ is a set of states with two distinguished subsets: $S_i \subseteq S$ (initial states), and $S_f \subseteq S$ (final states); for every state $s \in S$ is associated with a memory (an attribute) $a_s$ with its domain $A_s$;
- $T \subseteq S \times S$ is the set of transitions. Every transition $t = (s, s') \in T$ associated with

  **enabling predicate** $P_t : A_s \longrightarrow \mathbf{bool}$, and

  **transformational function** $f_t : A_s \longrightarrow A_{s'}$.

# Generalized Automaton

**A general automaton** is a system $\mathfrak{A} = (\mathbf{A}, A, \Box)$ where

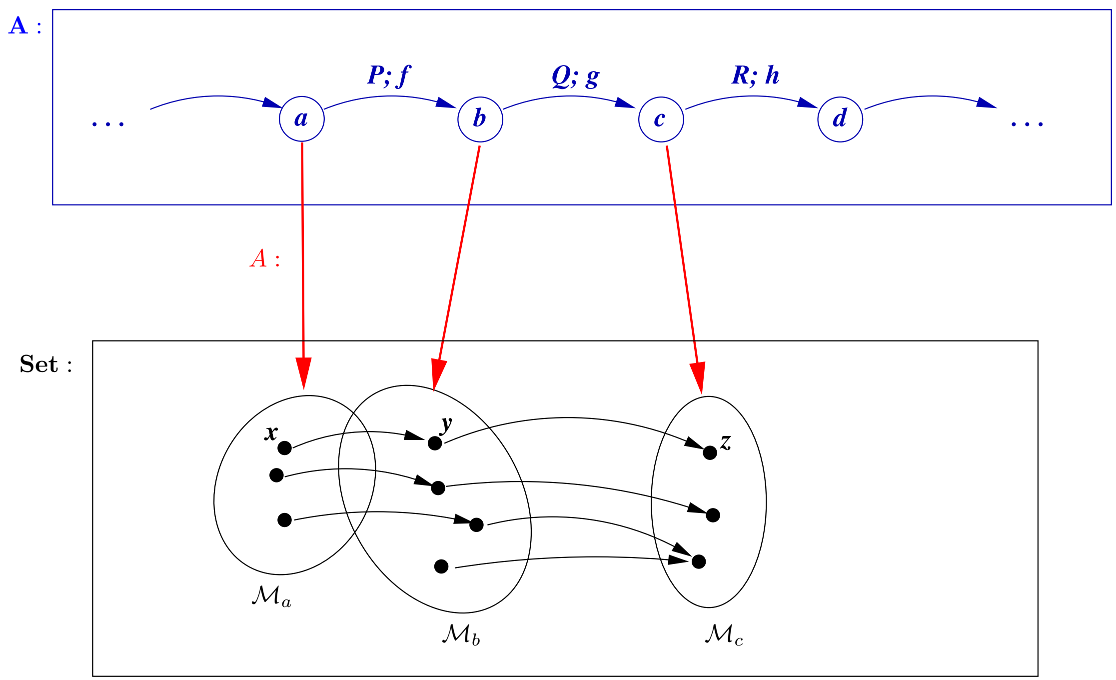
- $\mathbf{A}$ is a (small) category
- $A$ is a functor $\mathbf{A} \longrightarrow \mathbf{Set}$
- $\Box$ is a (partial) feedback operation

$$\Box : \left( \coprod_{a \in Obj(\mathbf{A})} a^A \times \mathrm{Mor}(\mathbf{A}) \right) \longrightarrow \mathrm{Mor}(\mathbf{A}),$$

satisfying the condition

$$x \Box (f \cdot g) = f^A(x) \Box g$$

# Generalized Automaton




$$\left.\begin{array}{rcl} g & = & x\Box f \\ h & = & y\Box g \\ y & = & f^A(x) \end{array}\right\} \Rightarrow x\,\Box(f\cdot g) = h = y\Box g = f^A(x)\Box g$$

# Wreath products

$$\left.\begin{array}{rcl}\mathfrak{A} & = & (\mathbf{A}, A, \square) \\ \mathfrak{B} & = & (\mathbf{B}, B, \boxdot)\end{array}\right\} \Rightarrow \mathfrak{C} = (\mathbf{C}, C, \boxtimes) = (\mathbf{A} \operatorname{wr} \mathbf{B},\ A \operatorname{wr} B,\ \boxtimes)$$

# Wreath products

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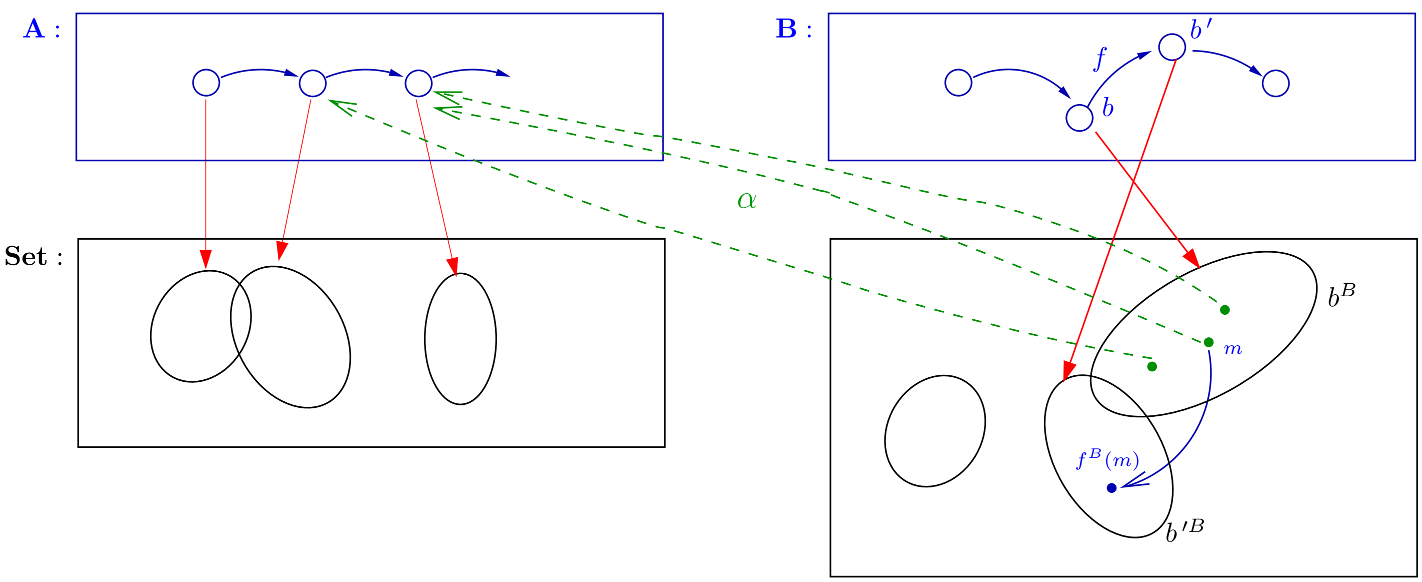
# *Wreath products of categories* (1)

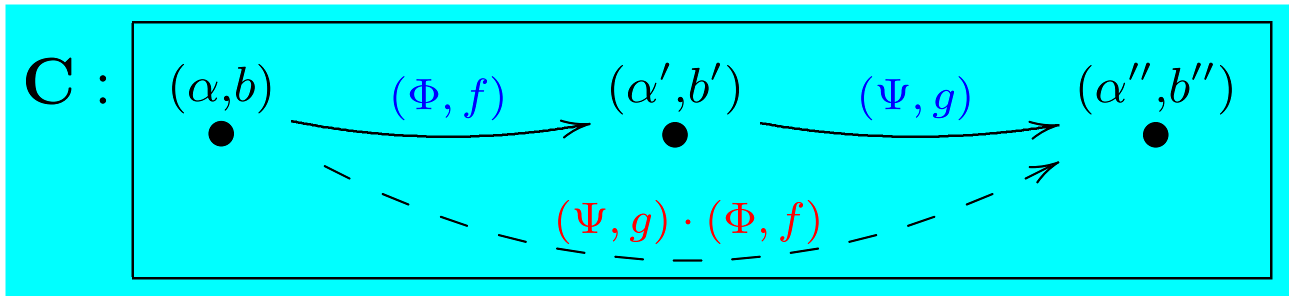
**1. $\mathbf{C} = \mathbf{A}\mathrm{wr}\mathbf{B}$**

$$\mathrm{Obj}(\mathbf{C}) \stackrel{\mathrm{def}}{=} \{(\alpha, b) \mid b \in \mathrm{Obj}(\mathbf{B}), \quad \alpha : b^B \longrightarrow \mathrm{Obj}(\mathbf{A})\}$$

$$\mathrm{Mor}_{\mathbf{C}}((\alpha, b), (\alpha', b')) \stackrel{\mathrm{def}}{=} \{(\Phi, f) \quad \mid \quad f \in \mathrm{Mor}_{\mathbf{B}}(b, b'), \quad \Phi = \bigcup_{m \in b^B} \mathrm{Mor}_{\mathbf{A}}(\alpha(m), \alpha'(f^B(m)))\}$$



# *Wreath products of categories* (2)




$(\Phi, f) \cdot (\Psi, g) = (\Phi * {}^{f}\Psi, f \cdot g)$, where the collection $\Phi * {}^{f}\Psi$ of morphisms in $\mathbf{A}$ is defined by the rule

$$\forall m \in b^{B}, \qquad (\Phi * {}^{f}\Psi)(m) = \Phi(m) \cdot \Psi(f^{B}(m)).$$

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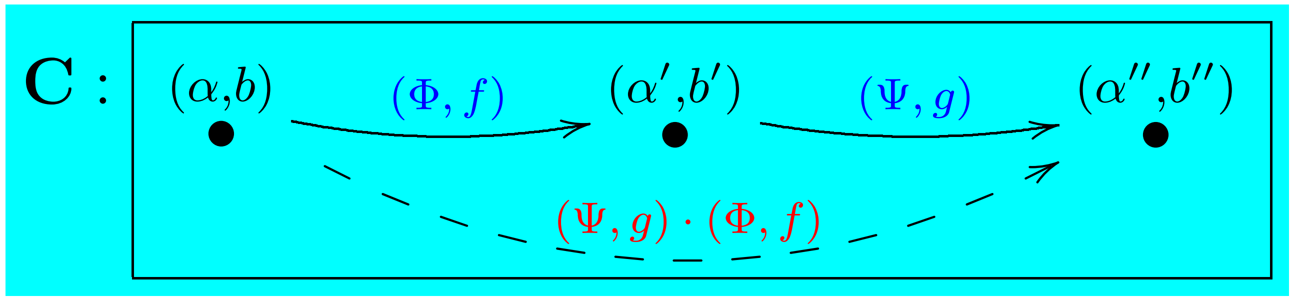


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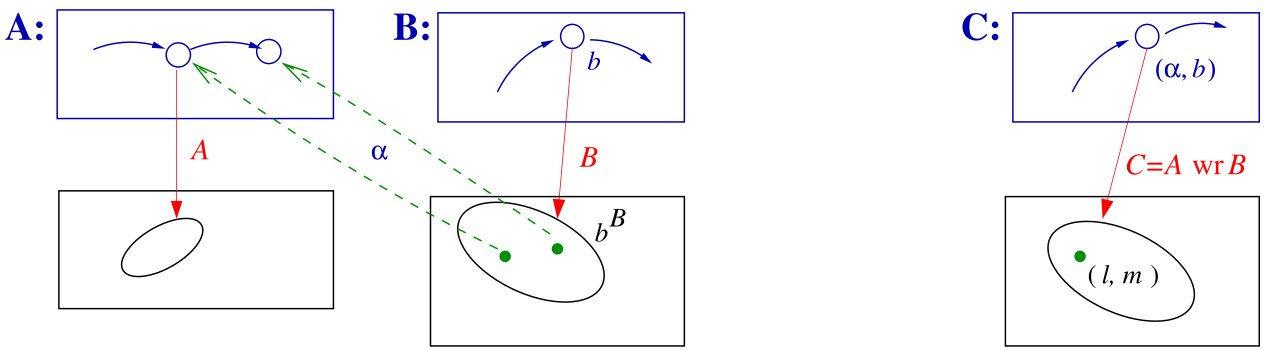
$$\forall m \in b^B, \qquad (\Phi * {}^f\Psi)(m) = \Phi(m) \cdot \Psi(f^B(m)).$$

**Lemma** $\mathbf{C}$ is a category.

# Wreath products of functors

$$\boxed{2. \quad C = A \,\mathrm{wr}\, B : \mathbf{A} \,\mathrm{wr}\, \mathbf{B} \longrightarrow \mathbf{Set}}$$




$$(\alpha, b)^C \stackrel{\text{def}}{=} \{(l, m) \mid m \in b^B, \quad l \in [\alpha(m)]^A\}$$
$$(\Phi, f)^C \stackrel{\text{def}}{=} (\Phi^A, f^B) \in \mathrm{Mor}_{\mathbf{Set}}((\alpha, b), (\alpha', b'))$$

# Wreath products of functors

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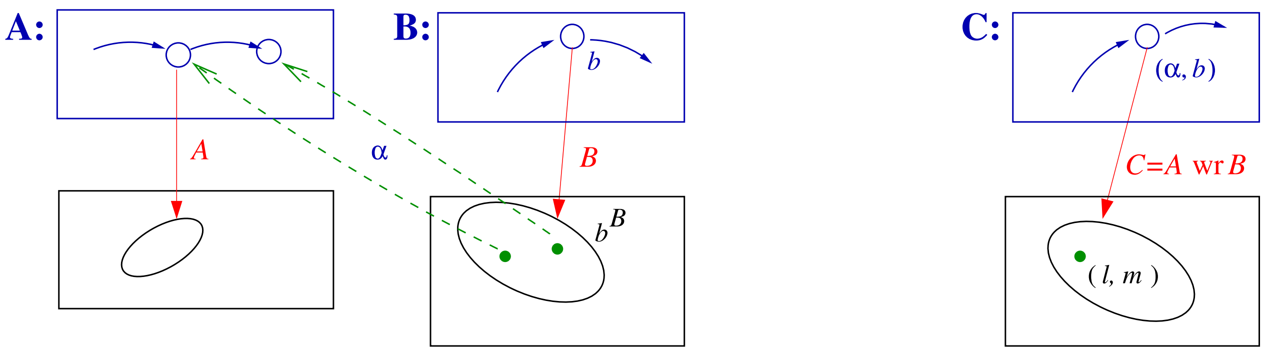


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**Lemma** $A \,\mathrm{wr}\, B$ is a functor from $\mathbf{C}$ to $\mathbf{Set}$.

# Composed control operator

$$3. \quad \boxtimes : (\coprod c^C \times \mathrm{Mor}(\mathbf{C})) \longrightarrow \mathrm{Mor}(\mathbf{C})$$

$$(l, m) \boxtimes (\Phi, f) \stackrel{\mathrm{def}}{=} (l \square \Phi(m), m \boxdot f)$$

**Lemma:** $(l, m) \boxtimes ((\Phi, f) \cdot (\Psi, g)) = (((\Phi, f)^{A wr B})(l, m)) \boxtimes (\Psi, g)$

# *Well-foundedness of the construction*

**Theorem.** $(\mathbf{C}, C, \boxtimes)$ is a generalized automaton

# Conclusions

- Categorical definition of generalized automata (GA) is given
- Wreath product of GAs is introduced and its correctness is shown
- Specialization of this wreath product give some intuitively simple (parallel, serial etc.) and more complex compositions of automata in progress.