A universal model for an infinitary version of sequential PCF

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Abstract

The aim of these notes is to give a universal model for the language $\mathrm{SPCF}_{\top}^{\infty}$ which is an infinitary version of sequential PCF. The universal model for this language resides inside the category of Laird domains which were introduced recently by Jim Laird in his paper "Bistability and Bisequentiality" for the sake of providing universal models for $\Lambda_{\perp}^{\top}$(simply typed λ-calculus over the Sierpinski-Space as ground type) and a fully abstract model for $\mathrm{SPCF}_{\top}$, a sequential extension of PCF with a non-recoverable error element $\top$, in a purely extensional way.

A Laird domain (or bistable biorder) is a triple $(B, \sqsubseteq, \leq)$ such that B is a set and $\sqsubseteq$ and $\leq$ are partial orders on B respecting some coherence conditions. We refer to $\sqsubseteq$ as the *extensional* and to $\leq$ as the *bistable* order. A function $f : (B, \sqsubseteq, \leq) \to (C, \sqsubseteq, \leq)$ is called *bistable* if it is monotonic with respect to both orders and stable and costable with respect to the bistable order. Bistable biorders and and bistable functions form a cartesian closed category named BB.

Jim Laird has shown that the category BB (more precisely the finite type hierarchy over the bistable biorder Σ which has two elements $\perp$ and $\top$ such that $\perp \sqsubseteq \top$ and $\perp \leq \top$) provides a universal model for $\Lambda_{\perp}^{\top}$. It turns out that the bistable biorders in the finite type hierarchy over Σ have an additional property, namely a well-behaved involution $\neg : B \to B$. We refer to a bistable biorder which has an involution as bbi. Each connected component with respect to the bistable order in a bbi forms a boolean lattice, and the involution acts on them like the boolean negation. It can be shown that also the full sub-category BBI of BB, i.e. bbis with bistable functions, is cartesian closed.

The original motivation in the definition of bistable biorders was to give a symmetric version of Berry's bidomains such that not only the supremum but also the infimum is ruled out. Having given a bbi one can define a stable and a costable order that the extensional and bistable orders can be recovered from them and further that a monotonic function between bbis is bistable if and only if it is stable (respectively costable) with respect to the stable (respectively costable) order. For all bbis $(B, \sqsubseteq, \leq)$ in the finite type hierarchy over

Σ the domains $(B, <_s)$ and $(B, >_c)$, where $<_s$ (respectively $<_c$) is the stable (respectively costable) order, are dI-domains.

In 1994 R. Cartwright, P.L. Curien and M. Felleisen constructed fully abstract models for the language SPCF, a sequential extension of PCF with catch operators and a set of error generators. Later, Curien described those models as the category OSA of sequential data structures with observably sequential algorithms/functions. Both Laird domains and sequential data structures provide a cpo-enriched fully abstract model of $\mathrm{SPCF}_\top$and thus are isomorphic. This isomorphism can be made explicit by defining a bbi from a sequential data structure via the stable and costable order and vice versa.

J. Longley showed in 1998 that in the category SA of sequential data structures with sequential algorithms/functions the sequential data structure $\mathbb{N} \to \mathbb{N}$ is universal. This result could be extended to the category OSA and also holds for the Laird domain model of $\mathrm{SPCF}_\top$.

Laird domains provide a universal model for $\Lambda_\perp^\top$and this result extends to binary SPCF (because the domain $\{t, f\}_\perp^\top$ is isomorphic to $(\Sigma \times \Sigma) \to \Sigma)$, but for proper SPCF, i.e. over $\mathbb{N}_\perp^\top$ the type of bilifted natural numbers, the model of Laird domains cannot be universal simply by cardinality reasons. But, if we allow for infinitary syntax, then the model of Laird domains turns out to be universal.

When solving the domain equations

$$D \simeq C \to R$$

and

$$C \simeq D \times C$$

or equivalently

$$D \simeq D^{\mathbb{N}} \to R$$

(which are a semantic reconstruction of Krivine's machine due to B. Reus and Th. Streicher using continuation semantics) in Laird domains then the Domain D satisfies the equation $D \simeq D^D$ and the ensuing model of untyped λ-calculus with an unrecoverable error-element $\top$ has the remarkable property that it is universal with respect to infinite λ-terms possibly containing $\top$.

The infinitary extension $\mathrm{SPCF}_\top^\infty$of $\mathrm{SPCF}_\top$is simply $\mathrm{SPCF}_\top$augmented by some new term constructors

$$M ::= \mathrm{SPCF}_\top\text{-constructors} \mid \lambda(x_i)_{i\in\omega}.t$$

and

$$t ::= M\langle M_i\rangle_{i\in\omega} \mid \top \mid \perp$$

where $\lambda(x_i)_{i\in\omega}.t$ respectively $M < M_i\rangle_{i\in\omega}$ are infinite abstraction respectively application. The terms of M are denotated in the domain D, while the elements of t are denotated in the domain R (of "results"), which will be in our case the domain Σ. Next, we consider the question, whether

$$D \simeq D^{\mathbb{N}} \to \Sigma$$

provides a universal model for the infinitary CPS target language CPS_∞ whose syntax is given by the following grammar:

$$M ::= x \mid \lambda(x_i)_{i\in\omega}.M\langle M_i\rangle_{i\in\omega}$$

We can show that the minimal solution of the above equation in Laird domains provides a universal model for CPS_∞. The proof is done by induction over the type structure of $\mathrm{SPCF}^\infty_\top$ by defining for each type σ bistable maps $e_\sigma : [\![\sigma]\!] \to D$ and $p_\sigma : D \to [\![\sigma]\!]$ with $p_\sigma \circ e_\sigma = \mathrm{id}_{[\![\sigma]\!]}$, such that for each context $\Gamma \equiv x_1 : \sigma_1, \ldots, x_n : \sigma_n$ and term-in-context $\Gamma \vdash t : \sigma$ there exists a CPS_∞ term M in the variables $\vec{x} \equiv x_1, \ldots, x_n$, such that the diagram

$$\begin{tikzcd} D^n \arrow[r, "{[\![\vec{x}\vdash M]\!]}"] \arrow[d, "r_\Gamma"'] & D \arrow[d, "r_\sigma"] \\ {[\![\Gamma]\!]} \arrow[r, "{[\![\Gamma\vdash t]\!]}"] & {[\![\sigma]\!]} \end{tikzcd}$$

commutes.

As Laird domains provide a universal model for CPS_∞ one could also ask if this model is also fully complete in the sense that two CPS_∞ terms M_1 and M_2 have the same infinite normal form if and only if their denotations in D are equal. This hope however has to be rejected, since for all terms M' the terms

$$M \equiv \lambda(x_i)_{i\in\omega}.x_0\langle\bot, \lambda(y_i)_{i\in\omega}.x_0\langle M', \bot, \bot, \ldots\rangle, \bot, \bot, \ldots\rangle$$

turn out to have the same denotation as the term

$$\lambda(x_i)_{i\in\omega}.x_0\langle\bot, \bot, \ldots\rangle$$

So, the separation theorem fails in CPS_∞.

Finally, an affine version of Laird domains can be used to give an extensional model of Girard's Ludics (more precisely of negative designs), which was one of the motivating questions for starting the work within Laird domains.