Infinite Trace Semantics

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1 Deterministic Language

We consider a simple call-by-name language of commands, with these terms:

$$N ::= \quad \mathtt{tick}.N \mid \mathtt{x} \mid \mu\mathtt{x}.N$$

The intended meaning of $\mathtt{tick}.M$ is "print a tick on the screen and then execute M". We write $\mathtt{diverge}$ for $\mu\mathtt{x}.\mathtt{x}$. We define a typing judgement

$$\mathtt{x}_0, \ldots, \mathtt{x}_{n-1} \vdash N \tag{1}$$

in the obvious way, and give an operational semantics on closed commands:

$$\begin{aligned} \mathtt{tick}.M &\rightsquigarrow^{\mathsf{tick}} M \\ \mu\mathtt{x}.M &\rightsquigarrow^{\tau} M[\mu\mathtt{x}.M/\mathtt{x}] \end{aligned}$$

where τ is a "silent" action. For each closed command M, there is a unique sequence

$$M = M_0 \rightsquigarrow^{a_0} M_1 \rightsquigarrow^{a_1} \cdots \tag{2}$$

Let Beh be the domain of vertical natural numbers. We write $[M] \in \mathsf{Beh}$ for the number of ticks in the sequence (2).

We define a denotational semantics in the standard way: a command (1) denotes a continuous function from Beh^n to Beh, and μ is interpreted as least fixpoint. For a closed command, $[\![M]\!] = [M]$.

2 Adding Erratic Choice

We now add a general erratic choice construct:

$$M ::= \quad \cdots | \mathtt{choose}\ i \in I.\ M_i$$

where $\{M_i\}_{i \in I}$ is a family of commands which can be finite, denumerable, continuum-size or empty. (The computational significance of these last two possibilities is a question we ignore here.) We add, for each $\hat{\imath} \in I$, the transition rule:

$$\mathtt{choose}\ i \in I.\ M_i \rightsquigarrow^{\tau} M_{\hat{\imath}}$$

Definition 1. *Let M be a closed command. An element $n \in \mathsf{Beh}$ is an* infinite trace *of M when there exists a sequence (2) with n ticks. We write $[M] \in \mathcal{P}\mathsf{Beh}$ (powerset, not powerdomain) for the set of infinite traces.*

Following Brookes, we define an *infinite trace semantics* to be one where $[\![M]\!] = [M]$ for each closed command M. In particular, the commands

```
diverge or tick.tick.diverge
diverge or tick.diverge or tick.tick.diverge
```

must have different denotations, by contrast with powerdomain/Roscoe semantics and game models in the literature.

Proposition 1. *1. For each $b \in \mathcal{P}\mathsf{Beh}$, there is a closed command M such that $[M] = b$. (This requires empty and continuum choice.)*
2. Given a command (1), and closed commands $M_0, \ldots, M_{n-1}$ and $M'_0, \ldots, M'_{n-1}$ such that $[M_0] = [M_1], \ldots, [M_{n-1}] = [M'_{n-1}]$, we have

$$[N[\overrightarrow{M_i/\mathtt{x}_i}]] = [N[\overrightarrow{M'_i/\mathtt{x}_i}]]$$

3 Environments

Definition 2. *A denotational model is* environmentally extensional *if a term's denotation is determined by the denotation of its closed substitution instances.*

This motivates the following:

Definition 3. *For each term (1), define the function*

$$(\mathcal{P}\mathsf{Beh})^n \xrightarrow{[N]} \mathcal{P}\mathsf{Beh}$$

$$\overrightarrow{[M_i]} \longmapsto [N[\overrightarrow{M_i/\mathtt{x}_i}]]$$

It is clear that if $\mathtt{x} \vdash N$ then $[\mu\mathtt{x}.N]$ is a fixpoint of the endofunction $[N]$. If we could say how to find this fixpoint, we would have a denotational semantics. However this cannot be done:

Proposition 2. *There is no environmentally extensional infinite trace semantics.*

Proof. Consider the following commands $\mathtt{x} \vdash N, N'$.

$$N = (\mathtt{choose}\ n \in \mathbb{N}.\ (\mathtt{tick})^n.\ \mathtt{diverge})\ \ \mathtt{or}\ \ \mathtt{x}$$
$$N' = (\mathtt{choose}\ n \in \mathbb{N}.\ (\mathtt{tick})^n.\ \mathtt{diverge})\ \ \mathtt{or}\ \ \mathtt{tick.x}$$

It can be seen that $[N]$ and $[N']$ are the same endofunction on $\mathcal{P}\mathsf{Beh}$, but $[\mu\mathtt{x}.N]$ and $[\mu\mathtt{x}.N']$ are different fixpoints of it.

If countable mutual recursion and only *binary* erratic choice are provided, then, as is well-known, we can encode a variant of `choose` $n \in \mathbb{N}$ that might diverge in the process of choosing. The example still stands if this is used.