An Analog Characterization of Computable Functions Over the Real Numbers

Olivier Bournez and Emmanuel Hainry

LORIA/INRIA, 615 Rue du Jardin Botanique
{Olivier.Bournez,Emmanuel.Hainry}@loria.fr

Abstract We present an *analog* and *machine-independent* algebraic characterization of elementary computable functions over the real numbers in the sense of recursive analysis: we prove that they correspond to the smallest set of functions that contains some basic functions, and closed by composition, linear integration, and a simple limit schema.
We generalize this result to all higher levels of the Grzegorczyk Hierarchy.
This paper improves several previous partial characterizations and has a dual interest:
– Concerning *recursive analysis*, it provides fully machine-independent characterizations of natural classes of computable functions over the real numbers, allowing to define these classes without usual considerations on higher-order/type 2 Turing machines.
– Concerning *analog models*, it provides a simple characterization of the power of a natural class of analog models over the real numbers, and provides new insights for understanding the relations between several analog computational models.

1 Introduction

Several approaches have been proposed to model computations over real numbers. *Recursive analysis* or *computable analysis*, was introduced by Turing [29], Grzegorczyk [12], Lacombe [16]. Alternative discrete-time computational models have also been investigated: see e.g. [4].

These models concern *discrete time* computability. Models of machines where the time is *continuous* can also be considered. The first ever built computers were continuous time machines: e.g. *Blaise Pascal's pascaline* or Lord Kelvin's model of *Differential Analyzer* [15], that gave birth to a real machine, built in 1931 at the MIT to solve differential equations [7], and which motivated the Shannon *General Purpose Analog Computer (GPAC) model* [27], whose computational power was characterized algebraically in terms of solutions of polynomial differential equations with real coefficients [27,23,26]. Continuous time machines also include analog neural networks [28], hybrid systems [3,5], or theoretical physic models [21,14,11]: see also survey [22].

The relations between all the models are not fully understood. One can say, that the theory of analog computations has not yet experienced the unification that digital discrete time computations have experienced through Turing work and the so-called *Church thesis* [9,22].

This however becomes a crucial matter since the progress of electronics makes the construction of some of the machines realistic, whereas some models were recently proved very (far too?) powerful: using the so-called *Zeno's paradox*, some models make it possible to compute non-Turing computable functions in a constant time: see e.g. [17,6,3,14,11].

In [17], Moore introduced a class of functions over the reals inspired from the classical characterization of computable functions over integers: observing that the continuous analog of a primitive recursion is a differential equation, Moore proposes to consider the class of $\mathbb{R}$-*recursive functions*, defined as the the smallest set of functions containing some basic functions, and closed by composition, differential equation solving (called *integration*), and minimization.

This class of functions, also investigated in [19,18], is closely related to GPAC computable functions, since $\mathbb{R}$-recursive functions that can be defined without minimization correspond to GPAC computable functions [17].

Putting aside possible objections about the physical feasibility of the μ-operator considered in paper [17], the original definitions of [17] suffer from several technical problems[1]. At least some of them make it possible to use a *"compression trick"* (another incarnation of Zeno's paradox) to simulate in a bounded time an unbounded number of discrete transitions in order to recognize arithmetical reals [17].

In his PhD dissertation, Campagnolo [9] proposes to restrict to the better-defined subclass $\mathcal{L}$ of $\mathbb{R}$-recursive functions corresponding to the smallest class of functions containing some basic functions and closed by composition and *linear* integration. Class $\mathcal{L}$ is related to functions elementary computable over integers in classical recursion theory and functions elementary computable over the real numbers in recursive analysis (discussed in [31]): any function of class $\mathcal{L}$ is elementary computable in the sense of recursive analysis, and conversely, any function over the integers computable in the sense of classical recursion theory is the restriction to integers of a function that belongs to $\mathcal{L}$ [9,8].

However, the previous results do not provide a characterization of *all* functions over the reals that are computable in the sense of recursive analysis. This paper provides one:

Theorem 1. *For functions over the reals of class $\mathcal{C}^2$ with a compact domain, f is elementary computable in the sense of recursive analysis iff it belongs to the smallest class of functions containing some basic functions and closed by composition, linear integration and a simple limit schema.*

We extend this theorem to a characterization of all higher levels of the Grzegorczyk hierarchy.

Theorem 2. *For functions over the reals of class $\mathcal{C}^2$ with a compact domain, f is computable in the sense of recursive analysis in level $n \geq 3$ of the Grzegorczyk hierarchy iff f belongs to the smallest class of functions containing some (other) basic functions and closed by composition, linear integration and a simple limit schema.*

Concerning *analog models*, these results have several impacts: first, they contribute to understand analog models, in particular the relations between GPAC computable functions, $\mathbb{R}$-recursive functions, and computable functions in the sense of recursive analysis. Furthermore, they prove that there cannot happen any *Super-Turing* phenomena for these classes of functions. In particular we have a "robust" class of functions in the sense of [13,2].

Concerning *recursive analysis*, our theorems provide *a purely algebraic* and *machine independent* characterization of elementary computable functions over the reals. Observe the potential benefits offered by these characterizations compared to classical definitions of these classes in recursive analysis, involving discussions about higher-order (type 2) Turing machines: see e.g. [30].

In Section 2, we start by some mathematical preliminaries. In Section 3, we recall some notions from classical recursion theory. We present basic definitions of recursive analysis in Section 4. Previous known results are recalled in Section 5. Our characterizations are presented in Section 6. The proofs are given in remaining sections.

For completeness, this submitted version has a very long appendix with all the proofs, plus an extension that shows that the limit schema can in some sense be avoided.

2 Mathematical preliminaries

Let $\mathbb{N}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{R}^{\geq 0}$ denote the set of integers, the set of rational numbers, the set of real numbers, and the set of non-negative real numbers respectively. Given $x \in \mathbb{R}^n$, we write $\overrightarrow{x}$ to emphasize that x is a vector.

We will use the following simple mathematical result[2].

[1] For example not well defined functions are considered, $\infty \times 0$ is always considered as 0, etc.... Some of them are discussed in [9,8] and even in the original paper [17].

[2] Proof is in appendix.

Lemma 1. *Let $F : \mathbb{R} \times \mathcal{V} \subset \mathbb{R}^2 \to \mathbb{R}^l$ be a function of class*[3] *$\mathcal{C}^1$, and $\beta(x) : \mathcal{V} \to \mathbb{R}$ be some continuous function. Assume that for all t and x, $\frac{\partial^2 F}{\partial t \partial x}(t,x)$ exists and $\|\frac{\partial F}{\partial t}(t,x)\| \leq K \exp(-t\beta(x))$, and $\|\frac{\partial^2 F}{\partial t \partial x}(t,x)\| \leq K \exp(-t\beta(x))$ for some constant $K > 0$.*

For all $x \in \mathcal{D}$, where $\mathcal{D}$ is the subset of the $x \in \mathcal{V}$ with $\beta(x) > 0$, $F(t,x)$ has a limit $L(x)$ in $t = +\infty$. Function $L(x)$ is of class $\mathcal{C}^1$, and its derivative L' is the limit of $\frac{\partial F(t,x)}{\partial x}(t,x)$ in $t = +\infty$. Furthermore

$$\|F(t,x) - L(x)\| \leq \frac{K\exp(-t\beta(x))}{\beta(x)} \quad \text{and} \quad \|\frac{\partial F}{\partial x}(t,x) - L'(x)\| \leq \frac{K\exp(-t\beta(x))}{\beta(x)}.$$

The following result[4], with previous lemma, is a key to provide upper bounds on the growth of functions of our classes (c.f. Lemma 4).

Lemma 2 (Bounding Lemma for Linear Differential Equations [1]). *For linear differential equation $\overrightarrow{x}' = A(t)\overrightarrow{x}$, if A is defined and continuous on interval $I = [a,b]$, where $a \leq 0 \leq b$, then, for all $\overrightarrow{x}_0$, the solution of $\overrightarrow{x}' = A(t)\overrightarrow{x}$ with initial condition $\overrightarrow{x}(0) = \overrightarrow{x}_0$ is defined and unique on I. Furthermore, the solution satisfies*

$$\|\overrightarrow{x}(t)\| \leq \|\overrightarrow{x}_0\| \exp(\sup_{\tau \in [0,t]} \|A(\tau)\| t).$$

3 Classical Recursion Theory

Classical recursion theory deals with functions over integers. Most classes of classical recursion theory can be characterized as closures of a set of basic functions by a finite number of basic rules to build new functions [25,20]: given a set $\mathcal{F}$ of functions and a set $\mathcal{O}$ of operators on functions (an operator is an operation that maps one or more functions to a new function), $[\mathcal{F}; \mathcal{O}]$ will denote the closure of $\mathcal{F}$ by $\mathcal{O}$.

Proposition 1 (Classical settings: see e.g. [25,20]). *Let f be a function from $\mathbb{N}^k$ to $\mathbb{N}^l$ for $k, l \in \mathbb{N}$. Function f is*

- *elementary iff it belongs to $\mathcal{E} = [0, S, U, +, \ominus; COMP, BSUM, BPROD]$;*
- *in class $\mathcal{E}_n$ of the Grzegorczyk Hierarchy ($n \geq 3$) iff it belongs to $\mathcal{E}_n = [0, S, U, +, \ominus, E_{n-1}; COMP, BSUM, BPROD]$;*
- *primitive recursive iff it belongs to $\mathcal{PR} = [0, U, S; COMP, REC]$;*
- *recursive iff it belongs to $\mathcal{R}ec = [0, U, S; COMP, REC, MU]$.*

The basic functions $0, (U_i^m)_{i,m \in \mathbb{N}}, S, +, \ominus$ and the operators $BSUM$, $BPROD$, $COMP$, REC, MU are given by

1. *$0 : \mathbb{N} \to \mathbb{N}$, $0 : n \mapsto 0$; $U_i^m : \mathbb{N}^m \to \mathbb{N}$, $U_i^m : (n_1, \dots, n_m) \mapsto n_i$; $S : \mathbb{N} \to \mathbb{N}, S : n \mapsto n+1$; $+ : \mathbb{N}^2 \to \mathbb{N}$, $+ : (n_1, n_2) \mapsto n_1 + n_2$; $\ominus : \mathbb{N}^2 \to \mathbb{N}$, $\ominus : (n_1, n_2) \mapsto max(0, n_1 - n_2)$;*
2. *$BSUM$: bounded sum. Given f, $h = BSUM(f)$ is defined by $h : (\overrightarrow{x}, y) \mapsto \sum_{z<y} f(\overrightarrow{x}, z)$; $BPROD$: bounded product. Given f, $h = BPROD(f)$ is defined by $h : (\overrightarrow{x}, y) \mapsto \prod_{z<y} f(\overrightarrow{x}, z)$;*
3. *$COMP$: composition. Given f and g, $h = COMP(f,g)$ is defined as the function verifying $h(\overrightarrow{x}) = g(f(\overrightarrow{x}))$;*
4. *REC : primitive recursion . Given f and g, $h = REC(f,g)$ is defined as the function verifying $h(\overrightarrow{x}, 0) = f(\overrightarrow{x})$ and $h(\overrightarrow{x}, n+1) = g(\overrightarrow{x}, n, h(\overrightarrow{x}, n))$.*
5. *MU : minimization. The minimization of f is $h : \overrightarrow{x} \mapsto \inf\{y; f(\overrightarrow{x}, y) = 0\}$.*

Functions E_n, involved in the definition of the classes $\mathcal{E}_n$ of the Grzegorczyk Hierarchy, are defined by induction as follows (when f is a function, $f^{[d]}$ denotes its d-th iterate: $f^{[0]}(\overrightarrow{x}) = x$, $f^{[d+1]}(\overrightarrow{x}) = f(f^{[d]}(\overrightarrow{x}))$):

1. *$E_0(x,y) = x + y$, $E_1(x) = (x+1) \times (y+1)$, $E_2(x) = 2^x$;*

[3] Recall that function $f : \mathcal{D} \subset \mathbb{R}^k \to \mathbb{R}^l$, $k, l \in \mathbb{N}$, is said of class $\mathcal{C}^r$ if it is r-times continuously differentiable on $\mathcal{D}$.

[4] As it was already the case in Campagnolo's Dissertation.

2. $E_{n+1}(x) = E_n^{[x]}(1)$ *for* $n \geq 2$.

We have $\mathcal{E} \subseteq \mathcal{PR} \subseteq \mathcal{R}ec$, and the inclusions are known to be strict [25,20]. It is also known that $\mathcal{E}_3 = \mathcal{E}$ and $\mathcal{PR} = \cup_i \mathcal{E}_i$ [25,20]. If $TIME(t)$ and $SPACE(t)$ denote the classes of functions that are computable with time and space t, then, for all $n \geq 3$, $\mathcal{E}_n = TIME(\mathcal{E}_n) = SPACE(\mathcal{E}_n)$, and $\mathcal{PR} = TIME(\mathcal{PR}) = SPACE(\mathcal{PR})$ [25,20]. $\mathcal{PR}$ corresponds to functions computable using *loop programs*. $\mathcal{E}$ corresponds to computable functions bounded by some iterate of the exponential function [25,20].

In classical computability, more general objects than functions over the integers can be considered, in particular functionals, i.e. functions $\Phi : \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^k \to \mathbb{N}^l$. A functional will be said *elementary* (respectively. $\mathcal{E}_n$, *primitive recursive*, *recursive*) when it belongs to the corresponding[5] class.

4 Computable Analysis

The idea sustaining *Computable analysis*, also called *recursive analysis*, is to define computable functions over real numbers by considering functionals over fast-converging sequences of rationals [29,16,12,30].

Formally, assume that a representation of rational numbers by integers is fixed: let $\nu_{\mathbb{Q}}(r)$ be the rational represented by integer r. A product $\mathcal{C} = [a_1, b_1] \times \ldots \times [a_k, b_k]$ of compact intervals with rational endpoints can be encoded by an integer $\nu(\mathcal{C})$ encoding the list $\langle \nu_{\mathbb{Q}}(a_1), \nu_{\mathbb{Q}}(b_1), \ldots, \nu_{\mathbb{Q}}(a_k), \nu_{\mathbb{Q}}(b_k) \rangle$.

A sequence of integers $(x_i) \in \mathbb{N}^{\mathbb{N}}$ *represents a real number* x if it converges quickly toward x (denoted by $(x_i) \rightsquigarrow x$) in the following sense: $\forall i, |\nu_{\mathbb{Q}}(x_i) - x| < \exp(-i)$.

Definition 1 (Recursive analysis [30]). *A function $f : \mathcal{D} \to \mathbb{R}$, where $\mathcal{D} \subset \mathbb{R}^k$ is a product of compact intervals, is said computable (in the sense of recursive analysis) if there exists a recursive functional Φ : $\mathbb{N}^{\mathbb{N}} \times \mathbb{N} \to \mathbb{N}$ such that for all $x \in \mathcal{D}$, for all sequence $X = (x_n) \in \mathbb{N}^{\mathbb{N}}$, we have $(\phi(X, j))_j \rightsquigarrow f(x)$ whenever $X \rightsquigarrow x$.*

A function $f : \mathcal{D} \to \mathcal{R}$, where $\mathcal{D} \subset \mathbb{R}^k$ is not necessarily compact, is said computable if there exists a recursive functional $\Phi : \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^2 \to \mathbb{N}$ such that for all product $\mathcal{C}$ of compact intervals included in $\mathcal{D}$, $\forall x \in \mathcal{C}$, we have $(\phi(X, \nu(\mathcal{C}), j))_j \rightsquigarrow f(x)$ whenever $X = (x_n) \rightsquigarrow x$. A function $f : \mathcal{D} \to \mathcal{R}^l$, with $l > 1$, is said computable if all its projections are.

A function f will be said *elementary* (respectively $\mathcal{E}_n$) computable whenever the corresponding functional Φ is. The class of elementary (respectively $\mathcal{E}_n$) computable functions over the reals will be denoted by $\mathcal{E}(\mathbb{R})$ (resp. $\mathcal{E}_n(\mathbb{R})$). Observe that elementary computable functions were discussed in [31].

5 Real-recursive and recursive functions

Following the original ideas from [17], but avoiding the minimization schema, Campagnolo proposed in [9] to consider the following class, built in analogy with elementary computable functions over the integers.

Definition 2 ([9,8]). *Let $\mathcal{L}$ be the class of functions $f : \mathbb{R}^k \to \mathbb{R}^l$, for some $k, l \in \mathbb{N}$, defined by $\mathcal{L} = [0, 1, -1, \pi, U, \theta_3; COMP, LI]$ where the basic functions 0, 1, -1, π, $(U_i^m)_{i,m \in \mathbb{N}}$, θ_3 and the schemata $COMP$ and LI are the following:*

1. *$0, 1, -1, \pi$ are the corresponding constant functions; $U_i^m : \mathbb{R}^m \to \mathbb{R}$ are, as in the classical settings, projections: $U_i^m : (x_1, \ldots, x_m) \mapsto x_i$;*
2. *$\theta_3 : \mathbb{R} \to \mathbb{R}$ is defined as $\theta_3 : x \mapsto x^3$ if $x \geq 0, 0$ otherwise.*

[5] Formally, a function f over the integers can be considered as functional $\overline{f} : (S, \overrightarrow{n}) \mapsto f(\overrightarrow{n})$. Similarly, an operator Op on functions $f_1, \ldots, f_m$ over the integers can be extended to $\overline{Op}(F_1, \ldots, F_m) : (S, \overrightarrow{n}) \mapsto Op(f_1(S, .), \ldots, f_m(S, .))(\overrightarrow{n})$. We will still (abusively) denote by $[f_1, \ldots, f_p; O_1, \ldots, O_q]$ for the smallest class of functionals that contains basic functions $\overline{f_1}, \ldots, \overline{f_p}$, plus the functional $Map : (S, n) \to S_n$, the nth element of sequence S, and which is closed by the operators $\overline{O_1}, \ldots, \overline{O_q}$. For example, a functional will be said elementary iff it belongs to $\mathcal{E} = [Map, \overline{0}, \overline{S}, \overline{U}, \overline{+}, \overline{\ominus}; \overline{COMP}, \overline{BSUM}, \overline{BPROD}]$.

3. *COMP: composition is defined as in the classical settings: Given f and g, $h = COMP(f,g)$ is the function verifying $h(\overrightarrow{x}) = g(f(\overrightarrow{x}))$;*
4. *LI: linear integration. From g and h, $LI(g,h)$ is the maximal solution of the linear differential equation $\frac{\partial f}{\partial y}(\overrightarrow{x},y) = h(\overrightarrow{x},y)f(\overrightarrow{x},y)$ with $f(\overrightarrow{x},0) = g(\overrightarrow{x})$.*
   In this schema, if g goes to $\mathbb{R}^n$, $f = LI(g,h)$ also goes to $\mathbb{R}^n$ and $h(\overrightarrow{x},y)$ is a $n \times n$ matrix with elements in $\mathcal{L}$.

Example 1. Functions sin, cos are in class $\mathcal{L}$ since they can be obtained using projections from function $S : t \mapsto (\sin(t), \cos(t))$, which can be in turn defined by a simple integration schema.

Class $\mathcal{L}$ includes common functions[6] like $+$,$-$,$\times$,exp, or $x \to r$ for all $r \in \mathbb{Q}$, but contains only total functions [8]:

Proposition 2 ([8]). *All functions from $\mathcal{L}$ are continuous, defined everywhere, and of class $\mathcal{C}^2$.*

The previous class can be partially related to class $\mathcal{E}(\mathbb{R})$: a real extension of $f : \mathbb{N}^k \to \mathbb{N}^l$ is a function $\tilde{f}$ from $\mathbb{R}^k$ to $\mathbb{R}^l$ whose restriction to $\mathbb{N}^k$ is f.

Proposition 3 ([9,8]).
1. *$\mathcal{L} \subset \mathcal{E}(\mathbb{R})$: any function from $\mathcal{L}$ is elementary computable over real numbers.*
2. *"$\mathcal{E} \subset \mathcal{L}$": any elementary computable function over the integers, has a real extension that belongs to $\mathcal{L}$.*

These results were generalized to all higher levels of the Grzegorczyk hierarchy: in the rest of this paper, for $n \geq 3$, let $\overline{E_n}$ denote a monotone real extension of the function $\exp_n$ over the integers defined inductively by $\exp_2(x) = \exp(x)$, $\exp_{i+1}(x) = \exp_i^{[x]}(1)$.

Proposition 4 ([9,8]). *Let $\mathcal{L}_n = [0, 1, -1, \pi, U, \theta_3, \overline{E_{n-1}}; COMP, LI]$.*
1. *$\mathcal{L}_n \subset \mathcal{E}_n(\mathbb{R})$: any function from $\mathcal{L}_n$ is $\mathcal{E}_n$-computable.*
2. *"$\mathcal{E}_n \subset \mathcal{L}_n$": any $\mathcal{E}_n$-computable function over the integers, has a real extension that belongs to $\mathcal{L}_n$.*

Although Propositions 3 and 4 give the inclusions $\mathcal{L} \subset \mathcal{E}(\mathbb{R})$ and $\mathcal{L}_n \subset \mathcal{E}_n(\mathbb{R})$, they fail to characterize completely $\mathcal{E}(\mathbb{R})$ and $\mathcal{E}_n(\mathbb{R})$: these inclusions are strict. Indeed, $x \mapsto 1/x$ is elementary computable while Proposition 2 says that all functions from $\mathcal{L}$ are defined everywhere. A similar argument works for $\mathcal{E}_n(\mathbb{R})$. We conjecture the inclusions to be strict even when restricting to total functions.

6 Real-recursive and recursive functions revisited

We now propose to consider new classes of functions that we will prove to correspond precisely to $\mathcal{E}(\mathbb{R})$ and $\mathcal{E}_n(\mathbb{R})$.

First, we modify a little bit the composition schema[7].

Definition 3 (COMP schema). *Given f,g, if there is a product of closed intervals[8] F with $Range(f) \subset F \subset Domain(g)$, then function $COMP(f,g)$ is defined. It is defined by $COMP(f,g) : \overrightarrow{x} \mapsto g(f(\overrightarrow{x}))$ on all $\overrightarrow{x}$ where $f(\overrightarrow{x})$ and $g(f(\overrightarrow{x}))$ exist.*

Now, we suggest to add a limit operator denoted by LIM, inspired by Lemma 1: a polynomial β over $\mathbb{R}$ is a function of the form $\beta : \mathbb{R} \to \mathbb{R}$, $\beta : x \mapsto \sum_{i=0}^n a_i x^i$ for some $a_0, \ldots, a_n \in \mathbb{R}$.

Definition 4 (LIM schema). *Let $f : \mathbb{R} \times \mathcal{D} \subset \mathbb{R}^2 \to \mathcal{I}$, and $\beta : \mathcal{D} \to \mathbb{R}$ a polynomial with the following hypothesis: there exists a constant K such that for all t,x, $\|\frac{\partial f}{\partial t}(t,x)\| \leq K\exp(-t\beta(x))$, $\frac{\partial^2 f}{\partial t \partial x}(t,x)$ exists, and $\|\frac{\partial^2 F}{\partial t \partial x}(t,x)\| \leq K\exp(-t\beta(x))$.*

Then, for all interval $I \subset \mathbb{R}$ on which $\beta(x) > 0$, the function $F : I \to \mathbb{R}$, with $F(x) = \lim_{t\to+\infty} f(t,x)$ is defined, under the condition that it is of class[9] $\mathcal{C}^2$.

[6] This is recalled in Lemma 6 in appendix.
[7] We suspect the composition schema from Proposition 1 to bring non elementary computable functions: see discussion in appendix.
[8] That can be $\mathbb{R}^k$ when g is total.
[9] If f is of class $\mathcal{C}^1$, function F exists and is at least of class $\mathcal{C}^1$ by Lemma 1.

We are ready to define our classes:

Definition 5 (Classes $\mathcal{L}^*$, $\mathcal{L}^*_n$). *The class $\mathcal{L}^*$, and $\mathcal{L}^*_n$, for $n \geq 3$, of functions from $\mathbb{R}^k$ to $\mathbb{R}^l$, for $k, l \in \mathbb{N}$, are following classes:*

- $\mathcal{L}^* = [0, 1, -1, U, \theta_3; COMP, LI, LIM]$.
- $\mathcal{L}^*_n = [0, 1, -1, U, \theta_3, \overline{E_{n-1}}; COMP, LI, LIM]$.

Remark 1. Compared to the classes of previous section, we do not require the constant function π as a basic function and we have a new schema LIM.

Example 2. Previous classes can easily[10] be shown stable by the primitive operator that sends a function f to its primitive $PRIM(f)$ equal to $\overrightarrow{0}$ at $\overrightarrow{0}$.

Class $\mathcal{L}^*$ also includes some non-total functions, in particular the function $\frac{1}{x} : \mathbb{R}^{\geq 0} \to \mathbb{R}$, $\frac{1}{x} : x \mapsto \frac{1}{x}$: indeed, $PRIM(\exp(-tx))$ is function $E : (t, x) \mapsto \frac{(1-\exp(-tx))}{x}$ for $x \neq 0$, t for $x = 0$ (of class $\mathcal{C}^k$ for all k). Now $\frac{1}{x} = LIM(E, id)$.

Proposition 5. *$\mathcal{L} \subset \mathcal{L}^*$, $\mathcal{L} \subset \mathcal{L}^*_n$ for all $n \geq 3$.*

Proof. The function $x \mapsto \pi$ is actually in $\mathcal{L}^*$. Indeed, from $x \mapsto \frac{1}{1+x^2}$ in the class[11], we have $\arctan x = PRIM(\frac{1}{1+x^2})$, and $\pi = \arctan(1)$. Observing that functions of $\mathcal{L}$ are total, and that our composition schema for total functions subsumes the composition schema of class $\mathcal{L}$, the result follows.

The main results of this paper are the following (proved in following two sections):

Theorem 1 (Characterization of $\mathcal{E}(\mathbb{R})$). *Let $f : \mathcal{D} \subset \mathbb{R}^k \to \mathbb{R}^l$ be some function over the reals of class $\mathcal{C}^2$, with $\mathcal{D}$ compact. f is in $\mathcal{E}(\mathbb{R})$ iff it belongs to $\mathcal{L}^*$.*

Theorem 2 (Characterization of $\mathcal{E}_n(\mathbb{R})$). *Let $f : \mathcal{D} \subset \mathbb{R}^k \to \mathbb{R}^l$ be some function over the reals of class $\mathcal{C}^2$, with $\mathcal{D}$ compact. Let $n \geq 3$. f is in $\mathcal{E}_n(\mathbb{R})$ iff it belongs to $\mathcal{L}^*_n$.*

7 Upper bounds

We now prove the upper bound $\mathcal{L}^* \subset \mathcal{E}(\mathbb{R})$. As one may expect, this sense of the proof has many similarities with the proof $\mathcal{L} \subset \mathcal{E}$ in [9,8]: main differences lie in the presence of non-total functions and of schema LIM.

A structural induction shows[12]:

Lemma 3. *All functions from $\mathcal{L}^*$ are of class $\mathcal{C}^2$ and defined on a domain of the form $I_1 \times I_2 \ldots \times I_k$ where each I_i is an interval.*

We propose to introduce the following notation: given $a \in \mathbb{R}$, let ρ_a be the function $x \mapsto \frac{1}{x-a}$. Let $\rho_{+\infty}$ and $\rho_{-\infty}$ be the function identity $x \mapsto x$.

Given $a, b \in \mathbb{R} \cup \{-\infty\} \cup \{+\infty\}$, for $I = (a, b)$, define $\rho_I(x) = |\rho_a(x)| + |\rho_b(x)|$. For $I = [a, b)$, define $\rho_I(x) = |\rho_b(x)| + |x|$. For $I = (a, b]$, define $\rho_I(x) = |\rho_a(x)| + |x|$. For $I = [a, b]$, define $\rho_I(x) = |x|$. For $\mathcal{D} = I_1 \times I_2 \ldots \times I_k$, let $\rho_{\mathcal{D}}(x) = \rho_{I_1}(x) + \ldots + \rho_{I_k}(x)$. In any case, $\rho_{\mathcal{D}}(x)$ is elementary computable and grows to $+\infty$ when x gets close to a bound of domain $\mathcal{D}$.

The following Lemma is an extension of a Lemma of [9,8] (it is proved by structural induction[13] using Lemma 1 for schema LIM, Lemma 2 for schema LI, plus the fact that by adjusting constants A, B it is always possible to assume that the degree of a product or a sum of two functions f and g is less than the maximum of their degrees).

[10] See Lemma 6 in appendix.
[11] See Lemma 6 in appendix.
[12] See full proof in appendix.
[13] See proof in appendix.

Lemma 4. *Let $f : \mathcal{D} \subset \mathbb{R}^k \to \mathbb{R}^l$ be a function of $\mathcal{L}^*$. There exist some integer d, and some constants A and B such that for all $\overrightarrow{x} \in \mathcal{D}$, $\|f(\overrightarrow{x})\| \leq A \exp^{[d]}(B\rho_{\mathcal{D}}(\overrightarrow{x}))$. Call the smallest such integer d the degree of f. All the partial derivatives of f also have a finite degree.*

We are ready to prove the upper bound.

Proposition 6. *$\mathcal{L}^* \subseteq \mathcal{E}(\mathbb{R})$.*

Proof.
- The basic functions $0, 1, -1, U, \theta_3$ are elementary computable.
- When $h = COMP(f, g)$, f and g elementary computable, then h is also elementary computable: indeed, there exists some closed set F with $Range(f) \subset F \subset Domain(g)$. Adapting the constructions in [30], given a product of compact intervals $\mathcal{C}$ included in $Domain(f)$, we can compute elementary a product of compact intervals $\mathcal{C}'$ with $f(\mathcal{C}) \subset \mathcal{C}'$. Now, for $x \in \mathcal{C}$, compose the functional that computes g on $\mathcal{C}' \cap F$ with the one that computes f on $\mathcal{C}$.
- Let $g = LIM(f, \beta)$, with f computed by elementary functional ϕ. We give the proof for f defined on $\mathbb{R} \times \mathcal{C}$ where $\mathcal{C}$ is a compact interval of $\mathbb{R}$. The general case is easy to obtain.
  Let $x \in \mathbb{R}$, with $\beta(x) > 0$. Since $\beta(x)$ is a polynomial, $1/\beta(x)$ can be bounded elementary by some computable integer N in some computable neighborhood of x.
  Let $(x_n) \rightsquigarrow x$. For all $i, j \in \mathbb{N}$, if we write (i) for the constant sequence $k \mapsto i$, we have $|\nu_{\mathbb{Q}}(\phi((i), (x_n), j)) - f(i, x)| < \exp(-j)$. By Lemma 1, we have $\|f(i, x) - g(x)\| \leq \frac{K \exp(-\beta(x)i)}{\beta(x)} \leq KN \exp(-\beta(x)i)$. Hence,
  $$|\nu_{\mathbb{Q}}(\phi((i), (x_n), j)) - g(x)| < \exp(-j) + KN \exp(-\beta(x)i).$$
  If we take $j' = j+1$, $i' = N(j+1+\lceil \ln(KN) \rceil)$, we have $exp(-j') \leq \exp(-j)/2$, and $KN \exp(-\beta(x)i') \leq \exp(-j)/2$. Hence g is computed by the functional $\psi : ((x_n), j) \mapsto \phi((N(j+1+\lceil \ln(KN) \rceil)), (x_n), j+1)$.
- Let $f = LI(g, h)$. The proof for this case is really the same as in [9,8].
  The idea is that, to find ϕ elementary computing f, one can use a numeric integration algorithm (Euler's Method). Using standard estimations on the error of a numeric Euler's method integration method (see e.g. [10]), the number of required steps can be bounded elementary, if one has an elementary bound on g, h and their derivatives: see[14] [9,8].

This ends the proof.

Replacing the bounds of Lemma 4 by bounds of type $\|f(\overrightarrow{x})\| \leq A\overline{E}_{n-1}^{[d]}(B\rho_{\mathcal{D}}(\overrightarrow{x}))$, in previous proofs yields the upper bounds for the levels of the *Grzegorczyk* hierarchy: for $n \geq 3$, $\mathcal{L}_n^* \subseteq \mathcal{E}_n(\mathbb{R})$.

8 Lower bounds

We will now consider the opposite inclusion: $\mathcal{E}(\mathbb{R}) \subseteq \mathcal{L}^*$, proved for functions of class $\mathcal{C}^2$ on compact domains.

Let $\epsilon > 0$ be some real. We write $\mathbb{N}\epsilon$ for the set of reals of the form $i\epsilon$ for some integer i. Given $y \in \mathbb{R}$, write $\lfloor y \rfloor_\epsilon$ for the unique $j\epsilon$ with j integer and $y \in [j\epsilon, j\epsilon + \epsilon)$.

Lemma 5. *Let $\epsilon : \mathbb{R} \to \mathbb{R}$ be some decreasing function, with $\epsilon(x) > 0$ for all x. Write ϵ_i for $\epsilon(\lfloor i \rfloor)$.*
Given $f : \mathbb{R}^2 \to \mathbb{R}^l$ in $\mathcal{L}^*$, there exists $F : \mathbb{R}^2 \to \mathbb{R}^l$ in $\mathcal{L}^*$ with the following properties:

- *For all $i \in \mathbb{N}$, $x \in \mathbb{N}\epsilon_i$, $F(i, x) = f(i, x)$*
- *For all $i \in \mathbb{N}$, $x \in \mathbb{R}$, $\|F(i, x) - f(i, \lfloor x \rfloor_{\epsilon_i})\| \leq \|f(i, \lfloor x \rfloor_{\epsilon_i} + \epsilon_i) - f(i, \lfloor x \rfloor_{\epsilon_i})\|$*
- *For all $i \in \mathbb{R}$, $x \in \mathbb{R}$, $\|\frac{\partial F}{\partial i}(i, x)\| \leq 5\|f(\lfloor i+1 \rfloor, \lfloor x \rfloor_{\epsilon_i}) - f(\lfloor i \rfloor, \lfloor x \rfloor_{\epsilon_i})\| + 25\|f(\lfloor i \rfloor, \lfloor x \rfloor_{\epsilon_i} + \epsilon_i) - f(\lfloor i \rfloor, \lfloor x \rfloor_{\epsilon_i})\| + 25\|f(\lfloor i+1 \rfloor, \lfloor x \rfloor_{\epsilon_{i+1}} + \epsilon_{i+1}) - f(\lfloor i+1 \rfloor, \lfloor x \rfloor_{\epsilon_{i+1}})\|$.*

[14] The full proof is in appendix.

Proof (Sketch[15]). Let $\zeta = \frac{3\pi}{2}$. Let $\gamma : x \mapsto \zeta\theta_3(\sin(2\pi x))$. $\forall i, \int_i^{i+1} \gamma = 1$ and γ is equal to 0 on $[i+\frac{1}{2}, i+1]$ for every $i \in \mathbb{N}$. Let $\Gamma = PRIM(\gamma)$ its primitive, and $int : x \mapsto \Gamma(x - \frac{1}{2})$. The function int is similar[16] to the integer part: $\forall i$, $\forall x \in [i, i+\frac{1}{2}]$, $int(x) = i = \lfloor x \rfloor$. Let $\Delta(i,x) = f(i, x+\epsilon(i)) - f(i,x)$.

For all i,x, we have

$$\begin{aligned} \frac{\gamma(x/\epsilon(i))}{\epsilon(i)}\Delta(i, \epsilon(i)\ int(x/\epsilon(i))) &= 0 \textit{ whenever } x - \lfloor x \rfloor_{\epsilon(i)} \geq \epsilon(i)/2 \\ &= \frac{\gamma(x/\epsilon(i))}{\epsilon(i)}\Delta(i, \lfloor x \rfloor_{\epsilon(i)}) \textit{ otherwise.} \end{aligned}$$

Let G be the solution of the linear differential equation $G(i,0) = f(i,0)$, $\frac{\partial G}{\partial x}(i,x) = \frac{\gamma(x/\epsilon(i))}{\epsilon(i)}\Delta(i, \epsilon(i) int(x/\epsilon(i)))$. An easy induction on j then shows that $G(i, j\epsilon(i)) = f(i, j\epsilon(i))$ for all integer j.

Then, let $\Delta'(i,x) = G(i+1,x) - G(i,x)$. We have

$$\begin{aligned} \gamma(i)\Delta'(int(i), x) &= 0 \textit{ whenever } i - \lfloor i \rfloor \geq 1/2 \\ &= \gamma(i)\Delta'(\lfloor i \rfloor, x) \textit{ otherwise} \end{aligned}$$

Let F be the solution of the linear differential equation $F(0,x) = G(0,x)$, $\frac{\partial F}{\partial i} = \gamma(i)\Delta'(int(i), x)$. By induction, we have $F(i,x) = f(i,x)$ for all $i \in \mathbb{N}$, $x \in \mathbb{N}\epsilon_i$.

Some technical computations[17] on the variations of function F allow to prove all the claims.

We are now ready to prove the missing sense of Theorem 1.

Proposition 7. *Let $f : \mathcal{D} \subset \mathbb{R}^k \to \mathbb{R}^l$ be some function over the reals of class $\mathcal{C}^2$, with $\mathcal{D}$ compact. If f is $\mathcal{E}(\mathbb{R})$, then it belongs to $\mathcal{L}^*$.*

Proof. We give the proof for a function f defined on interval $[0,1]$ to $\mathbb{R}$. The general case is easy to obtain.

Since f'' is continuous on a compact set, f'' is bounded by some constant M. By mean value theorem, we have $|f'(x) - f'(y)| \leq M|x-y|$ for all x, y.

Given i, consider[18] n with $\exp(-i)e^n \geq M$. For all j, consider $x_j = \frac{j}{e^n}$, so that for all $x, y \in [x_j, x_{j+1}]$ we have $|f'(x) - f'(y)| \leq \exp(-i)$.

For all j, let y_j be some rational number at most $\exp(-i)$ far from $f(x_j)$ and $z_j = \frac{y_{j+1} - y_j}{e^n}$. By mean value theorem, $z_j = f'(\chi_j)$ for some $\chi_j \in [x_j, x_{j+1}]$, which implies $|f'(\chi_j) - f'(x_j)| < \exp(-i)$, and so, z_j is at most $\exp(-i)$ far from $f'(x_j)$.

Writing $z_j = \frac{p_j}{q_j}$ for some integers p_j and q_j, and observing that the y_j, and so the z_j can be elementary obtained from i and j, the functions $p_{\mathbb{N}} : \mathbb{N}^2 \to \mathbb{N}$, and $q_{\mathbb{N}} : \mathbb{N}^2 \to \mathbb{N}$ that map (i,j) to corresponding p_j and q_j are elementary computable. By Proposition 3, they can be extended to function $p : \mathbb{R}^2 \to \mathbb{R}$ and $q : \mathbb{R}^2 \to \mathbb{R}$ in $\mathcal{L}$. Consider function $g : \mathbb{R} \times [0,1] \to \mathbb{R}$ defined on all $i, x \in \mathbb{R}$ by $g(i,x) = p(i, e^n x)/q(i, e^n x)$. By construction, for i, j integer, we have $g(i, x_j) = z_j$.

Consider the function F given by Lemma 5 for function g and $\epsilon : n \mapsto \exp(-n)$. We have $F(i, x_j) = z_j$ for all $i, j \in \mathbb{N}$.

For all integer i, and all $x \in \mathbb{R}$, we have

$$\begin{aligned} \|F(i,x) - f'(x)\| &\leq \|F(i,x) - F(i, \lfloor x \rfloor_\epsilon)\| + \|F(i, \lfloor x \rfloor_\epsilon - g(i, \lfloor x \rfloor_\epsilon)\| \\ &\quad + \|g(i, \lfloor x \rfloor_\epsilon) - f'(\lfloor x \rfloor_\epsilon)\| + \|f'(\lfloor x \rfloor_\epsilon) - f'(x)\| \\ &\leq \|F(i, \lfloor x \rfloor_\epsilon + \epsilon_i) - F(i, \lfloor x \rfloor_\epsilon)\| + 0 + \exp(-i) + M\epsilon_i \\ &\leq \|z_{j+1} - z_j\| + 2\exp(-i) \\ &\leq \|f'(x_{j+1}) - z_{j+1}\| + \|f'(x_j) - z_j\| + \|f'(x_{j+1}) - f'(x_j)\| \\ &\quad + 2\exp(-i) \\ &\leq 5\exp(-i) \end{aligned}$$

[15] The full proof is in appendix.
[16] See graphical representations of γ and int in Figures 1 and 2 in appendix.
[17] Detailed in appendix.
[18] We voluntarily sometimes also write e^n for $\exp(n)$.

Consider the function $G : \mathbb{R}^2 \to \mathbb{R}$ defined for all $i, x \in \mathbb{R}$ by the linear differential equation $G(i, 0) = f(0)$[19] and $\frac{\partial G}{\partial x}(i, x) = F(i, x)$. For all integer i, we have $G(i, 0) - f(0) = 0$ and $\|\frac{\partial G}{\partial x}(i, x) - f'(x)\| = \|F(i, x) - f'(x)\| \leq 5\exp(-i)$. By mean value theorem on function $G(i, x) - f(x)$, we get $\|G(i, x) - f(x)\| \leq 5\exp(-i)$ on $[0, 1]$.

Hence, $f(x)$ is the limit of $G(i, x)$ when i goes to $+\infty$ with integer values. We just need to check that schema LIM can be applied to function G of $\mathcal{L}^*$ to conclude: indeed, the limit of $G(i, x)$ when i goes to $+\infty$ will exist and coincide with this value, i.e. $f(x)$.

Since $\frac{\partial G}{\partial x} = F$, and hence $\|\frac{\partial^2 G}{\partial i \partial x}\| = \|\frac{\partial F}{\partial i}\|$ and since $\frac{\partial G}{\partial i} = \int_0^x \frac{\partial F}{\partial i}(i, x)dx$ implies $\|\frac{\partial G}{\partial i}\| \leq \int_0^1 \|\frac{\partial F}{\partial i}\|dx \leq \|\frac{\partial F}{\partial i}\|$ we only need to prove that we can bound $\|\frac{\partial F}{\partial i}\|$ by $K\exp(-i)$ for a constant K.

But from Lemma 5, we know that for all i, x, $\|\frac{\partial F}{\partial i}(i, x)\| \leq 5\|g(\lfloor i+1 \rfloor, \lfloor x \rfloor_{\epsilon_i}) - g(\lfloor i \rfloor, \lfloor x \rfloor_{\epsilon_i})\| + 25\|g(\lfloor i \rfloor, \lfloor x \rfloor_{\epsilon_i} + \epsilon_i) - g(\lfloor i \rfloor, \lfloor x \rfloor_{\epsilon_i})\| + 25\|g(\lfloor i+1 \rfloor, \lfloor x \rfloor_{\epsilon_{i+1}} + \epsilon_{i+1}) - g(\lfloor i+1 \rfloor, \lfloor x \rfloor_{\epsilon_{i+1}})\|$.

First term can be bounded by $5\exp(-i) + 5\exp(-i) = 10\exp(-i)$.

Second term can be bounded by $25(\|g(\lfloor i \rfloor, \lfloor x \rfloor_{\epsilon_i} + \epsilon_i) - f'(\lfloor x \rfloor_{\epsilon_i} + \epsilon_i)\| + \|f'(\lfloor x \rfloor_{\epsilon_i} + \epsilon_i) - f'(\lfloor x \rfloor_{\epsilon_i})\| + \|g(\lfloor i \rfloor, \lfloor x \rfloor_{\epsilon_i}) - f'(\lfloor x \rfloor_{\epsilon_i})\|) \leq 25\exp(-i) + 25\exp(-i) + 25\exp(-i) = 75\exp(-i)$.

Similarly for third term, replacing i by $i + 1$.

Hence $\|\frac{\partial F}{\partial i}(i, x)\| \leq 160\exp(-i)$, and so schema LIM can be applied on function G of $\mathcal{L}^*$ to get function f. This ends the proof.

The missing sense of Theorem 2 can be proved similarly for all levels $n \geq 3$ of the Grzegorczyk hierarchy. Furthermore, we have.

Proposition 8. *Let $f : \mathcal{D} \subset \mathbb{R}^k \to \mathbb{R}^l$ be some function over the reals of class $\mathcal{C}^2$, with $\mathcal{D}$ compact. If f is $\mathcal{E}_n(\mathbb{R})$, for $n \geq 3$, then it belongs to $\mathcal{L}_n^*$.*

Remark 2.
- We have actually a *normal form theorem*: previous proof shows that every function of $\mathcal{L}^*$ and $\mathcal{L}_n^*$ can be defined using only 2 schemata LIM. One to get function $\frac{1}{x}$ plus the one of the proof.
- A corollary of this remark is that composing several LIM schemata is always equivalent to at most two for functions of our classes.

References

1. V. I. Arnold. *Ordinary Differential Equations.* MIT Press, 1978.
2. E. Asarin and A. Bouajjani. Perturbed Turing machines and hybrid systems. In *Logic in Computer Science*, pages 269–278, 2001.
3. E. Asarin and O. Maler. Achilles and the tortoise climbing up the arithmetical hierarchy. *Journal of Computer and System Sciences*, 57(3):389–398, dec 1998.
4. L. Blum, F. Cucker, M. Shub, and S. Smale. *Complexity and Real Computation.* Springer-Verlag, 1998.
5. O. Bournez. Achilles and the Tortoise climbing up the hyper-arithmetical hierarchy. *Theoretical Computer Science*, 210(1):21–71, 6 1999.
6. O. Bournez. *Complexité Algorithmique des Systèmes Dynamiques Continus et Hybrides.* PhD thesis, Ecole Normale Supérieure de Lyon, Janvier 1999.
7. M. Bowles. United states technological enthusiasm and the british technological skepticism in the age of the analog brain. In *IEEE Annals of the History of Computing*, volume 4, pages 5–15, 1996.
8. M. Campagnolo, C. Moore, and J. F. Costa. An analog characterization of the Grzegorczyk hierarchy. *Journal of Complexity*, in print.
9. M. L. Campagnolo. *Computational complexity of real valued recursive functions and analog circuits.* PhD thesis, Universidade Technica de Lisboa, 2001.
10. J.-P. Demailly. *Analyse Numérique et Equations Différentielles.* Presses Universitaires de Grenoble, 1991.
11. G. Etesi and I. Németi. Non-Turing computations via Malament-Hogarth space-times. *International Journal Theoretical Physics*, 41:341–370, 2002.
12. A. Grzegorczyk. Computable functionals. *Fundamenta Mathematicae*, 42:168–202, 1955.

[19] A technique similar to the one we use here to get function f', can be used to show that $f(0)$ is always in $\mathcal{L}^*$.

13. T. Henzinger and J.-F. Raskin. Robust undecidability of timed and hybrid systems. *Hybrid systems: computation and control; Second International Workshop, HSCC'99, Berg en Dal, The Netherlands, March 29–31, 1999; proceedings*, 1569, 1999.
14. M. L. Hogarth. Does general relativity allow an observer to view an eternity in a finite time? *Foundations of Physics Letters*, 5:173–181, 1992.
15. W. T. Thomson (L. Kelvin). On an instrument for calculating the integral of the product of two given functions. In *Proceedings of the Royal Society of London*, number 24, pages 266–276, 1876.
16. D. Lacombe. Extension de la notion de fonction récursive aux fonctions d'une ou plusieurs variables réelles iii. *Comptes Rendus de l'Académie des Sciences Paris*, 241:151–153, 1955.
17. C. Moore. Recursion theory on the reals and continuous-time computation. *Theoretical Computer Science*, 162(1):23–44, 5 1996.
18. J. Mycka. Infinite limits and R-recursive functions. *Acta Cybernetica*, 16:83–91, 2003.
19. J. Mycka. μ-recursion and infinite limits. *Theoretical Computer Science*, 302:123–133, 2003.
20. P. Odifreddi. *Classical Recursion Theory*, volume 125 of *Studies in Logic and the foundations of mathematics*. North-Holland, APR 1992.
21. T. Ord. Hypercomputation: Computing more than the Turing machine. Technical report, University of Melbourne, September 2002. Available at http://www.arxiv.org/abs/math.LO/0209332.
22. P. Orponen. *Algorithms, Languages and Complexity*, chapter A survey of continuous-time computational theory, pages 209–224. Kluwer Academic Publishers, 1997.
23. M. B. Pour-El. Abstract computability and its relation to the general purpose analog computer (some connections between logic, differential equations and analog computers). *Transactions of the American Mathematical Society*, 199:1–28, 1974.
24. E. Ramis, C. Deschamp, and J. Odoux. *Cours de Mathématiques Spéciales, Tome 3, Topologie et éléments d'analyse*. Masson, FEB 1995.
25. H. Rose. *Subrecursion: Functions and Hierarchies*. Clarendon Press, 1984.
26. L. A. Rubel. Some mathematical limitations of the general-purpose analog computer. *Advances in Applied Mathematics*, 9:22–34, 1988.
27. C. E. Shannon. Mathematical theory of the differential analyser. *Journal of Mathematics and Physics MIT*, 20:337–354, 1941.
28. H. Siegelmann. *Neural Networks and Analog Computation - Beyond the Turing Limit*. Birkauser.
29. A. Turing. On computable numbers, with an application to the "Entscheidungsproblem". In *Proceedings of the London Mathematical Society*, volume 2, pages 230–265, 1936.
30. K. Weihrauch. *Computable Analysis*. Springer, 2000.
31. Q. Zhou. Subclasses of computable real valued functions. *Lecture Notes in Computer Science*, 1276:156–165, 1997.

9 Extension: Limit schema can be avoided.

Actually, there is no need to add explicitly a limit schema to class $\mathcal{L}$ to get a full characterization of $\mathcal{E}(\mathbb{R})$.

The point, is on the definition of what is called a *maximal* solution of a differential equation (see the definition of LI schema in Definition 2).

Indeed, in classical mathematical textbooks, solutions of differential equations are usually assumed to be defined on open sets. Indeed, if we focus to the case we are interested in, i.e. linear differential equations, a solution of linear differential equation $x' = A(t)x$, is a differentiable function $x : \mathcal{U} \subset \mathbb{R} \to \mathbb{R}^n$, where $\mathcal{U}$ is an *open set*, on which $A(t)$ is defined, that satisfies $x'(t) = A(t)x(t)$ for all $t \in \mathcal{U}$ [1,10]. The solution is said to be *maximal*, if there is no bigger domain $\mathcal{U}'$ on which it holds [1,10]. For linear differential equations, $\mathcal{U}$ must be an interval (and by previous definition an open interval) [1,10].

But consider the case, where this interval is left-bounded: $\mathcal{U} = (a, b)$, where $a \in \mathbb{R}$, $b \in \mathbb{R} \cup \{+\infty\}$. It might happen, that $A(t)$ is either defined in a, or has a limit A in a. In that latter case, we may denote $A(a)$ for the limit A (i.e. extend A by continuity).

By continuity of $A(t)x(t)$, for $t > a$, function $x'(t)$ has a limit l when t goes to a (this limit is $l = A(a)x(a)$). A priori, x is defined only on $(a, b]$. But by a well-known result[20] [24], since $x'(t)$ has a limit in $t = a$, actually x can be extended to a function $\overline{x}$ defined on whole $[a, b)$, of class $\mathcal{C}^1$, right derivable in a, with right derivative l. In other words, there exists a function $\overline{x} : [a, b) \to \mathbb{R}^n$, that satisfies

$$\overline{x}'(t) = A(t)x(t) \text{ for all } t \in [a, b) \tag{1}$$

where $\overline{x}'$ designates derivative for $t > a$, and (as it may seem natural considering the domain of $\overline{x}$) right derivative when $t = a$.

We believe that it is natural to call such a function *the maximal solution* of the linear differential equation.

Doing so, leads to propose the following schema, which is very similar to LI schema (it does not include LI, since we assume $\mathcal{C}$ compact).

Definition 6 ($\overline{LI}$ Schema). *Given g, and h defined on $\mathcal{D} \times \mathcal{C} \to \mathbb{R}^n$, with $\mathcal{D} \subset \mathbb{R}$, $\mathcal{C}$ compact, function $\overline{LI}(g, h)$ is a function f of class $\mathcal{C}^2$, defined on $\mathcal{V} \times \mathcal{C} \subset \mathbb{R}^2 \to \mathbb{R}^n$, where $\mathcal{V}$ is an interval with $0 \in \mathcal{V}$, with h either defined on $\mathcal{V} \times \mathcal{C}$ or with a $\mathcal{C}^1$ extension to this domain (still denoted abusively by h), such that*

1. *$f(\overrightarrow{x}, 0) = g(\overrightarrow{x})$, for all $\overrightarrow{x} \in \mathcal{C}$;*
2. *$f'(\overrightarrow{x}, t) = h(\overrightarrow{x}, t) f(\overrightarrow{x}, t)$ for all $t \in \mathcal{V}$, $x \in \mathcal{C}$.*

where $f'(\overrightarrow{x}, t)$ designates derivative relative to t when t is the interior of $\mathcal{V}$ and directional derivatives on the bounds of $\mathcal{V}$.

Consider then the following classes.

Definition 7. *The class $\mathcal{L}^{\overline{LI}}$ and $\mathcal{L}_n^{\overline{LI}}$ for $n \geq 3$, of functions from $\mathbb{R}^k$ to $\mathbb{R}^l$, for $k, l \in \mathbb{N}$, are the following classes:*

1. $\mathcal{L}^{\overline{LI}} = [0, 1, -1, \pi, U, \theta_3, \frac{1}{x}, \frac{1}{1-x}; COMP, LI, \overline{LI}]$.
2. $\mathcal{L}_n^{\overline{LI}} = [0, 1, -1, \pi, U, \theta_3, \frac{1}{x}, \frac{1}{1-x}, \overline{E_{n-1}}; COMP, LI, \overline{LI}]$.

We have:

Theorem 3. *Let $f : \mathcal{D} \subset \mathbb{R}^k \to \mathbb{R}^l$ be some function over the reals of class $\mathcal{C}^2$, with $\mathcal{D}$ compact.*

1. *f is in $\mathcal{E}(\mathbb{R})$ iff it belongs to $\mathcal{L}^{\overline{LI}}$.*
2. *f is in $\mathcal{E}_n(\mathbb{R})$ iff it belongs to $\mathcal{L}_n^{\overline{LI}}$.*

[20] Lemma 7 in appendix.

Proof. For the upper bound, the idea is as before to prove that basic functions are elementary (respectively $\mathcal{E}_n$) computable and that the schemata preserve elementary (respectively $\mathcal{E}_n$) computability. This was already proved for all schemata but schema $\overline{LI}$. But, Lemma 9 in appendix proves that this case is OK.

For the lower bound, consider the function $G(i,x)$ built in the proof of proposition 7, which is defined on (a superset of) $[0;+\infty[\times\mathcal{D}$. We have $\lim_{i\to\infty} G(i,x) = f(x)$, and $\|\frac{\partial G}{\partial i}\| \leq K\exp(-i)$ and $\|\frac{\partial^2 G}{\partial i \partial x}\| \leq K\exp(-i)$ for some constant K.

The idea is now to consider a suitable variable substitution: consider the function $\check{G}$ defined as $\check{G}(i,x) = G(-2\ln(1-i),x)$. We have $\lim_{i\to 1}\check{G}(i,x) = f(x)$.

Observe that $-2\ln(1-i)$ is in our class, since it can be obtained from $PRIM(\frac{1}{1-x})$, and that function $\check{G}$, from previous constructions, is in our class $\mathcal{L}^{\overline{LI}}$, and solution of a linear differential equation:

$$\begin{aligned}\tfrac{\partial \check{G}}{\partial i}(i,x) = \tfrac{2\gamma(-2\ln(1-i))}{1-i}\int_0^x & \big(g(int(-2\ln(1-i)),0) - g(int(-2\ln(1-i))+1,0)\\ & + \int_0^t \tfrac{\gamma(t(1-i)^2)}{(1-i)^2} g(-2\ln(1-i), t+(1-i)^2)\ dt\\ & - \int_0^t \tfrac{\gamma(t(1-i)^2)}{(1-i)^2} g(-2\ln(1-i), t)\ dt\big)\ dx\end{aligned}$$

where g is the function of the proof of Proposition 7 in $\mathcal{L}$, hence also in $\mathcal{L}^{\overline{LI}}$.

We have

$$\frac{\partial \check{G}}{\partial i}(i,x) = \frac{2}{1-i}\frac{\partial G}{\partial i}(-2\ln(1-i),x).$$

Hence

$$\|\frac{\partial \check{G}}{\partial i}(i,x)\| \leq \frac{2}{1-i}K\exp(2\ln(1-i)) \leq K(1-i).$$

This means that $\frac{\partial \check{G}}{\partial i}(i,x)$ has a limit when i goes to 1 for all x (observe that the convergence is uniform in x).

We also have

$$\frac{\partial^2 \check{G}}{\partial i \partial x}(i,x) = \frac{2}{1-i}\frac{\partial^2 G}{\partial i \partial x}(-2\ln(1-i),x).$$

Hence

$$\|\frac{\partial^2 \check{G}}{\partial i \partial x}(i,x)\| \leq \frac{2}{1-i}K\exp(2\ln(1-i)) \leq K(1-i).$$

This means that $\frac{\partial \check{G}}{\partial x}(i,x)$ converges uniformly in x when i goes 1.

By Lemma 8, $\check{G}$ can be extended to a function $\overline{\check{G}}$ of class $\mathcal{C}^1$ defined on the bound 1: this function is the *maximal* solution of the previous linear differential equation. f is then given by $f(x) = \overline{\check{G}}(1,x)$.

10 Appendix

10.1 Proof of Lemma 1

Proof (Of Lemma 1). By mean value theorem,

$$\|F(t,x)-F(t',x)\| \leq \int_t^{t'} K\exp(-t\gamma(x)) \leq K\int_t^{+\infty} \exp(-t\gamma(x)) = K\frac{\exp(-t\gamma(x))}{\gamma(x)}.$$

This implies that $F(t,x)$ satisfies Cauchy criterion, and hence converges in $t=+\infty$. Second item is obtained by letting t' go to $+\infty$. Third item is obtained by replacing $F(t,x)$ by $\frac{\partial F}{\partial x}(t,x)$ in previous upper bound. Observe that it implies that the convergence is uniform in x in every compact on which $\gamma(x)\geq\epsilon$ for some ϵ, in particular given x on the compact defined by the set of x' with $\gamma(x')\geq\gamma(x)/2$. The derivative of a converging sequence of functions, whose sequence of derivatives converges uniformly, exists and is the limit of the derivatives. Furthermore, the limit of a uniformly converging sequence of continuous functions is continuous.

10.2 Some examples of functions from classes $\mathcal{L}$ and $\mathcal{L}^*$

Lemma 6.
- *Functions* $id : x\mapsto x$, sin, cos, exp, $+,\times$, $x\mapsto 1+x^2$, $x\mapsto r$ *for all rational* q, *are in* $\mathcal{L}$ *and* $\mathcal{L}^*$.
- *Given* $f\in\mathcal{L}$ *or* $\mathcal{L}^*$ *its primitive function* F *equal to* $\overrightarrow{0}$ *at* $\overrightarrow{0}$, *denoted by* $PRIM(f)$ *also belongs to the same class.*

Proof. Function id is given by $LI(1,0)$. Function $S : t\mapsto(\sin(t),\cos(t))$ can be defined by $LI\left(\begin{pmatrix}0\\1\end{pmatrix},\begin{pmatrix}0&1\\-1&0\end{pmatrix}\right)$. Project this function on each of its two variables to get sinus and co-sinus function. Function exp is given by $LI(id,1)$. $+:\mathbb{R}^2\to\mathbb{R}$ is given by $x+0=x$, $\frac{\partial x+y}{\partial y}=1$. $\times:\mathbb{R}^2\to\mathbb{R}$ is given by $x\times 0=x$, $\frac{\partial x\times y}{\partial y}=x$. $x\mapsto 1+x^2$ is given by $1+x\times x$. $PRIM(f)$ can be defined by $\begin{pmatrix}F\\1\end{pmatrix}=LI\left(\begin{pmatrix}0\\1\end{pmatrix},x\mapsto\begin{pmatrix}0&f\\0&0\end{pmatrix}\right)$. Given $p,q\in\mathbb{N}$ with $q>0$, Function $x\mapsto p$, is $1+1+\ldots+1$, function $x\mapsto x^{q-1}$ is $x\times\ldots\times x$, and $PRIM(x\mapsto x^{q-1})$ is $x\mapsto x^q/q$ whose value in 1 is $1/q$. Hence $x\mapsto p/q=p\times 1/q$ is in $\mathcal{L}$.

10.3 Is class $\mathcal{E}_c$ stable by composition?

Question 1. Is the class $\mathcal{E}c$ of elementary computable functions over the reals stable by composition?

In fact, the problem may arise when considering functions defined on an open interval: consider a function f elementary computable that decreases to 0 faster than any function of the form $x\mapsto 1/\exp^{[d]}(x)$. The $\frac{1}{x}$ function is elementary computable. But the composition of $\frac{1}{x}$ and f grows faster than any iterated exponential and hence is not elementary computable.

10.4 Proof of Lemma 3

Proof (Of Lemma 3).

- This is clear for basic functions (1, 0, -1, U, and θ_3).
- Composition preserves this property.
- Linear differential equations preserve class $\mathcal{C}^2$ [1,10]. They also preserve the domain property: see Lemma 2 or [1,10].
- If $g=LIM(f,\gamma)$, from definition of LIM schema, this is clear.

10.5 Proof of Lemma 4

Proof (Of Lemma 4).

Let $\deg f$ denote the degree of function f.

By some elementary algebra and elementary properties of the exponential function, observe that by adjusting constants A, B, it is always possible to assume for all function f and g, $\deg fg \leq max(\deg f, \deg g)$, and $\deg(f+g) \leq max(\deg f, \deg g)$.

Now, by structural induction:

- 0, 1, -1, U and all their derivatives have degree at most 1.
- $\theta_3(x)$ and its derivative have degree 1.
- The degree of $COMP(f,g)$ is less than $deg(f) + deg(g)$, since $deg(f \circ g) \leq deg(f) + deg(g)$ can easily be established using basic properties of exponential function. By the chain rule, the degree of any of the derivative of the composition $f(g)$ is bounded by $\max_i(\deg \frac{\partial g}{\partial i}, \deg \frac{\partial f}{\partial i} + \deg g)$.
- For $f = LI(g,h)$ as in Definition 2, Lemma 2 allows to write

  $$\|f(x,y)\| \leq \|g(x)\| \exp(\sup_{\tau \in [0,y]} \|h(x,\tau)\| y).$$

  It follows that the degree of f is less than $\max(\deg g, \deg h + 1)$.

  The derivative of f relative to y is $h(\overrightarrow{x}, y) f(\overrightarrow{x}, y)$. Hence its degree is also bounded by $\max(\deg g, \deg h + 1)$. By [1,10], we know that the other derivative relative to variable x is solution of linear differential equation $d' = hd + \frac{\partial h}{\partial i} f$ with initial condition $d(x,0) = \frac{\partial g}{\partial x}$. The bound given by Lemma 2 for this linear differential equation allows to state that the degree of this derivative is less than $\max(\deg \frac{\partial g}{\partial x}, \deg h + 1, \deg \frac{\partial h}{\partial x} + 1, \deg f + 1)$.
- Let $g = LIM(f,\gamma)$ as in Definition 4. By Lemma 1, we know that $g(x) = \lim_{i \to \infty} f(i,x)$, g is of class $\mathcal{C}^1$, $\|g(x)\| \leq \|f(0,x)\| + K/\gamma(x)$ and $\|\frac{\partial g}{\partial x}\| \leq \|\frac{\partial f}{\partial x}(0,x)\| + K/\gamma(x)$. Now, the degree of $\frac{1}{\gamma(x)}$ for any polynomial γ can easily be shown to be less than 1. Hence, the degree g and of $\frac{\partial g}{\partial x}$ is less than the degree of f.

10.6 Proof of Lemma 5

Proof. Let $\zeta = \frac{3\pi}{2}$. Let $\gamma : x \mapsto \zeta\theta_3(\sin(2\pi x))$. $\forall i, \int_i^{i+1} \gamma = 1$ and γ is equal to 0 on $[i + \frac{1}{2}, i+1]$ for $i \in \mathbb{N}$. Let $\Gamma = PRIM(\gamma)$ its primitive, and $int : x \mapsto \Gamma(x - \frac{1}{2})$. int is a function similar to the integer part: $\forall i$, $\forall x \in [i, i + \frac{1}{2}]$, $int(x) = i = \lfloor x \rfloor$. Figure 1 and 2 are graphical representations of γ and int respectively.

Let $\Delta(i,x) = f(i, x + \epsilon(i)) - f(i,x)$. For all i,x, we have

$$\begin{aligned} \tfrac{\gamma(x/\epsilon(i))}{\epsilon(i)} \Delta(i, \epsilon(i)\ int(x/\epsilon(i))) &= 0 \ \textit{whenever}\ x - \lfloor x \rfloor_{\epsilon(i)} \geq \epsilon(i)/2 \\ &= \tfrac{\gamma(x/\epsilon(i))}{\epsilon(i)} \Delta(i, \lfloor x \rfloor_{\epsilon(i)})\ \textit{otherwise.} \end{aligned}$$

Let G be the solution of the linear differential equation $G(i,0) = f(0)$, $\frac{\partial G}{\partial x}(i,x) = \frac{\gamma(x/\epsilon(i))}{\epsilon(i)} \Delta(i, \epsilon(i) int(x/\epsilon(i)))$.

An easy induction on j then shows that $G(i, j\epsilon(i)) = f(i, j\epsilon(i))$ for all integer j.

Now, on $[j\epsilon(i), j\epsilon(i) + \epsilon(i))$, $G(i,x) - f(i, \lfloor x \rfloor_{\epsilon(i)}) = \int_{j\epsilon(i)}^{x - j\epsilon(i)} \frac{\gamma(x/\epsilon(i))}{\epsilon(i)} \Delta(i, \lfloor x \rfloor_{\epsilon(i)})$, hence, for all $i \in \mathbb{N}$, $\|G(i,x) - f(i, \lfloor x \rfloor_{\epsilon_i})\| \leq \|\Delta(i, \lfloor x \rfloor_{\epsilon_i})\| = \|f(i, \lfloor x \rfloor_{\epsilon_i} + \epsilon_i) - f(i, \lfloor x \rfloor_{\epsilon_i})\|$.

Now, let $\Delta'(i,x) = G(i+1,x) - G(i,x)$. For all i,x we have

$$\begin{aligned} \gamma(i)\Delta'(int(i),x) &= 0 \ \textit{whenever}\ i - \lfloor i \rfloor \geq 1/2 \\ &= \gamma(i)\Delta'(\lfloor i \rfloor, x)\ \textit{otherwise} \end{aligned}$$

Let F be the solution of linear differential equation $F(0,x) = G(0,x)$, $\frac{\partial F}{\partial i} = \gamma(i)\Delta'(int(i),x)$.

An easy induction on i shows that $F(i,x) = G(i,x)$ for all integer i, and all $x \in \mathbb{R}$. Hence $F(i,x) = f(i,x)$ for all $i \in \mathbb{N}$, $x \in \mathbb{N}\epsilon_i$ and $\|F(i,x) - f(i, \lfloor x \rfloor_{\epsilon_i})\| \leq \|f(i, \lfloor x \rfloor_{\epsilon_i} + \epsilon_i) - f(i, \lfloor x \rfloor_{\epsilon_i})\|$ for all $i \in \mathbb{N}$, $x \in \mathbb{R}$.

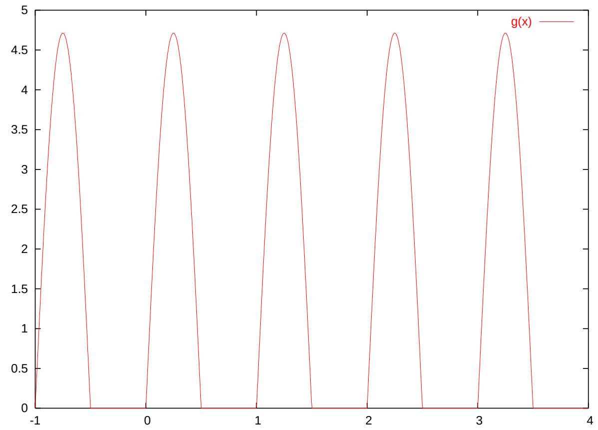

Figure 1. Graphical representation of γ

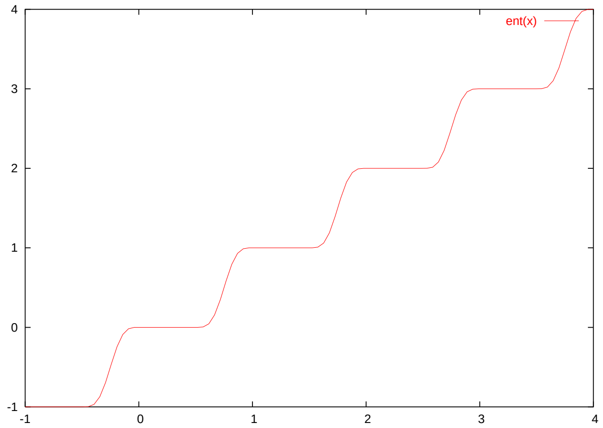

Figure 2. Graphical representation of int

Now, $\frac{\partial F}{\partial i}$ is either 0 or $\gamma(i)\Delta'(\lfloor i\rfloor, x) = \gamma(i)(G(\lfloor i+1\rfloor, x) - G(\lfloor i\rfloor, x))$. In any case, it is derivable in x, and hence $\frac{\partial^2 F}{\partial x \partial i}$ is either 0 or $\gamma(i)(\frac{\partial G}{\partial x}(\lfloor i+1\rfloor, x) - \frac{\partial G}{\partial x}(\lfloor i\rfloor, x))$.

When $x \in \mathbb{N}\epsilon_i$, bounding γ by 5 ($\zeta \leq 5$), $\|\frac{\partial F}{\partial i}\| \leq 5\|f(\lfloor i+1\rfloor, x) - f(\lfloor i\rfloor, x)\|$.

When $x \in \mathbb{R}$, $\|\frac{\partial^2 F}{\partial x \partial i}\| \leq \|\frac{\partial G}{\partial x}(\lfloor i{+}1\rfloor, x\| {+} \|\frac{\partial G}{\partial x}(\lfloor i\rfloor, x)\|$. The second term is either 0 or $5\|\frac{\gamma(x/\epsilon_i)}{\epsilon_i}\Delta(\lfloor i\rfloor, \lfloor x\rfloor_{\epsilon_i})\| \leq \frac{25}{\epsilon_i}\|\Delta(\lfloor i\rfloor, \lfloor x\rfloor_{\epsilon_i})\| = \frac{25}{\epsilon_i}\|f(\lfloor i\rfloor, \lfloor x\rfloor_{\epsilon_i} + \epsilon_i) - f(\lfloor i\rfloor, \lfloor x\rfloor_{\epsilon_i})\|$. A similar bound holds for first term, replacing i by $i+1$.

Using mean value theorem, $\|\frac{\partial F}{\partial i}(i,x)\| \leq \|\frac{\partial F}{\partial i}(i, \lfloor x\rfloor_{\epsilon_i}\| + \|\frac{\partial^2 F}{\partial x \partial i}(i,x)\|(x - \lfloor x\rfloor_{\epsilon_i}) \leq \|\frac{\partial F}{\partial i}(i, \lfloor x\rfloor_{\epsilon_i}\| + \epsilon(i)\|\frac{\partial^2 F}{\partial x \partial i}(i,x)\|$, which yields the expected bound.

10.7 Proof of Proposition 6: Case $f = LI(g,h)$

Proof (Case $f = LI(g,h)$ in the proof of Proposition 6).

Let $f = LI(g,h)$. We give the proof for $g : [0,1] \to \mathbb{R}$ and $h : [0,1] \times [c,d] \to \mathbb{R}$. The general case is easy to obtain.

This proof is copied from [9,8]. The idea is that, to find ϕ elementary computing f, one uses a numeric integration algorithm (Euler's Method).

First, let us note that f is twice differentiable with respect to its second variable since its derivative is the product of f and h that are differentiable. To compute $f(x,y)$, we will slice $[0,y]$ into segments of length λ and compute approximations of $f(x,\tau_i)$ for τ_i multiple of λ.

$h \in \mathcal{E}(\mathbb{R})$. Let ϕ_h computing h. Let $(\phi) \rightsquigarrow (x, \tau_i)$. Let us define $\gamma_i = \frac{(\phi_h(\phi))_n}{n+1}$ for n to be chosen.

$f \in \mathcal{E}(\mathbb{R})$. Let ϕ_g computing g. Let $(\phi_x) \rightsquigarrow x$. We will approach $f(x,\tau_i)$ by ψ_i defined by

$$\psi_0 = \frac{(\phi_g(\phi_x))_m}{m+1}$$

$$\psi_{i+1} = \psi_i + \lambda\psi_i\gamma_i$$

Let us now compute the error induced by our approximation. Let $\varepsilon_i = f(x,\tau_i) - \psi_i$.

$\forall i, \exists \chi \in [\tau_i, \tau_{i+1}]; f(x,\tau_{i+1}) = f(x,\tau_i) + \lambda f(x,\tau_i)h(x,\tau_i) + \frac{\lambda^2}{2}\frac{\partial f}{\partial y}(x,\chi)$.

$$\begin{aligned}
\varepsilon_{i+1} &= f(x,\tau_i) - \psi_i + \lambda f(x,\tau_i)h(x,\tau_i) - \lambda\psi_i\gamma_i + \tfrac{\lambda^2}{2}\tfrac{\partial f}{\partial y}(x,\chi) \\
|\varepsilon_{i+1}| &\leqslant |\varepsilon_i| + |\lambda h(x,\tau_i)(f(x,\tau_i) - \psi_i)| + |\lambda\psi_i(\gamma_i - h(x,\tau_i))| + |\tfrac{\lambda^2}{2}\tfrac{\partial f}{\partial y}(x,\chi)| \\
&\leqslant |\varepsilon_i| \times |1 + \lambda h(x,\tau_i)| + \lambda\psi_i|\gamma_i - h(x,\tau_i)| + \tfrac{\lambda^2}{2}\beta \\
&< |\varepsilon_i| \times |1 + \lambda h(x,\tau_i)| + \lambda\psi_i\tfrac{1}{n+1} + \tfrac{\lambda^2}{2}\beta
\end{aligned}$$

With $\beta = \max_{\chi \in [0,y]}(\frac{\partial f}{\partial y}(x,\chi))$.

$$\varepsilon_{i+1} < |\varepsilon_i| \times |1 + \lambda\overline{y}| + \tfrac{\lambda}{n+1}\overline{y} + \tfrac{\lambda^2}{2}\beta$$

With $\overline{y}$ set as a bound for h that can be elementary computed as shown by the preceding lemma.

Some little algebra shows then

$$\begin{aligned}
|\varepsilon_i| &< |\varepsilon_0|\,[1 + \lambda\overline{y}]^i + \left(\lambda\overline{y}\tfrac{1}{n+1} + \tfrac{\lambda^2}{2}\beta\right)\tfrac{(1+\lambda\overline{y})^i - 1}{\lambda\overline{y}} \\
&< \left[\tfrac{1}{m+1} + \tfrac{\lambda\beta}{2\overline{y}} + \tfrac{\psi_i}{\overline{y}(n+1)}\right]\exp(\lambda i\overline{y}) \\
&< \left[\tfrac{1}{m+1} + \tfrac{1}{n+1} + \tfrac{\lambda\beta}{2\overline{y}}\right]\exp(\lambda i\overline{y})
\end{aligned}$$

So, if we choose m, n, and i adequately (this choice can be made elementary), we can make the error as little as wanted. This proves that f is elementary computable and terminates our proof.

10.8 Extending a function

Lemma 7 ([24]). *Let $F : (a, b) \to \mathbb{R}^l$ be a function of class $\mathcal{C}^1$. Assume that its derivative F' has a limit l in b. Then*

1. *F has a limit (say L) in b.*
2. *F can be extended to a function $\overline{F} : (a, b] \to \mathbb{R}^l$ of class $\mathcal{C}^1$ with $\overline{F}(b) = L$.*
3. *The (left) derivative of $\overline{F}$ in b is l.*

Proof. Let (e_n) be a decreasing sequence converging to 0. For all $p > n$, by mean value theorem $\|F(b - e_n) - F(b - e_p)\| < |e_n - e_p| sup_{(b-e_n;b-e_p)}|F'|$ that converges to 0. Hence F satisfies Cauchy criterion and hence must have a limit in b.

Let $\overline{F}$ be defined by $\overline{F} = F$ on (a, b), and $\overline{F}(b) = L$. Function $\overline{F}$ has a left derivative in b of value $\lim_b \overline{F}'$. Indeed for all η, there exists $\epsilon > 0$ such that for all $x \in (b - \epsilon, b)$, $\|\overline{F}(x) - l\| \leq \eta$. Mean value theorem says that $\|\overline{F}(b - \epsilon) - \overline{F}(b)\| \leq \epsilon \sup_{(b-\epsilon,b)} \|\overline{F}'\|$ so $\|(\overline{F}(b - \epsilon) - \overline{F}(b))/\epsilon - l\| < \eta$, and hence $\overline{F}$ has a left derivative in b, whose value is l.

As a consequence, for functions of several variables, we get:

Lemma 8 (Extending a two variables function). *Let $F : (a, b) \times \mathcal{D} \to \mathbb{R}^l$, be a function of class $\mathcal{C}^1$, with $\mathcal{D} \subset \mathbb{R}$. Assume that, for all $x \in \mathcal{D}$, its first partial derivative $\frac{\partial F}{\partial t}(t, x)$ has a limit $l(x)$ when t goes to b. Assume that $\frac{\partial F}{\partial x}(t, x)$ converges uniformly in x when t goes to b.*

Then there exists a function $\overline{F} : (a, b] \to \mathbb{R}^l$ of class $\mathcal{C}^1$ that extends function F: $\overline{F}(t, x) = F(t, x)$ for all $t \in (a, b)$, $x \in \mathcal{D}$.

Furthermore, for all $x \in \mathcal{D}$, the left derivative of $\overline{F}(., x)$ in $t = b$ is $l(x)$.

Proof. Fixing x, previous lemma applied to $F(., x)$ implies the existence of $L(x) = \lim_{t=b} F(t, x)$. If we define $\overline{F}(b, x) = L(x)$ and $\overline{F}(t, x) = F(t, x)$ for $t < b$, we also know that the left derivative of $\overline{F}(., x)$ is $l(x)$ in $t = b$ for all x.

Since F is $\mathcal{C}^1$, all its partial derivatives exist and are continuous on $[a, b) \times \mathcal{D}$.

If $\frac{\partial F}{\partial x}(t, x)$ converges uniformly, since the limit of a converging sequence of derivable functions whose derivatives converge uniformly is derivable and has a derivative equal to the limit of the derivatives, and since the limit of a uniformly converging sequence of continuous functions is continuous, we know that L is derivable and continuous in all $x \in \mathcal{D}$ of derivative $\lim_{t=b} \frac{\partial F}{\partial x}(t, x)$. Having both partial derivatives $\frac{\partial \overline{F}}{\partial t}(., x)$ and $\frac{\partial \overline{F}}{\partial x}(b, .)$ continuous is not enough for guaranteeing that $\overline{F}$ is of class $\mathcal{C}^1$ (see [24]). We also need to observe that partial derivative on x is continuous on $(a, b] \times \mathcal{D}$: one needs the continuity of all partial derivatives but one [24].

But uniform convergence gives the continuity: indeed, $\|\frac{\partial \overline{F}}{\partial x}(b, x) - \frac{\partial \overline{F}}{\partial x}(t', x')\| \leq \|\frac{\partial \overline{F}}{\partial x}(b, x) - \frac{\partial F}{\partial x}(t, x)\| + \|\frac{\partial F}{\partial x}(t, x) - \frac{\partial F}{\partial x}(t, x')\|$ where first term can be put arbitrary close to 0 uniformly in x, and fixing $t = t_0$, the second can be put arbitrary close to 0.

Concerning the property of being elementary computable:

Lemma 9 (Extending a function from $\mathcal{E}(\mathbb{R})$). *Assume that $f : [a, b) \times \mathcal{D} \subset \mathbb{R}^k \to \mathbb{R}^l$ is a function of $\mathcal{E}(\mathbb{R})$. Assume that $\mathcal{D}$ is compact. Assume that we know that there exists a function $\overline{f} : [a, b] \times \mathcal{D}$ of class $\mathcal{C}^1$ that extends f: $\overline{f}(t, x) = f(t, x)$ for all $t \in (a, b)$, $x \in \mathcal{D}$.*

Then $\overline{f}$ is also in $\mathcal{E}(\mathbb{R})$.

Proof. We do the proof for $a = 0, b = 1$, $l = 1$ and $\mathcal{D} \subset \mathbb{R}$. The general case is easy to obtain. The derivative $\frac{\partial \overline{f}}{\partial x}$ is continuous on compact $[\frac{1}{2}, 1] \times \mathcal{D}$, hence bounded by some constant M. By mean value theorem, we know that $|\overline{f}(t, x) - \overline{f}(t', x)| \leq M|t - t'|$ for all t, t', x.

Now, given (t, x) and $(t_i, x_i) \rightsquigarrow (t, x)$, and given n, a rational number r with $|r - \overline{f}(t, x)| \leq \exp(-n)$ can be found as follows: compute i_0 so that $\exp(-i_0) \leq \frac{\exp(-n-1)}{3M}$.

Now, if $t_{i_0} - \exp(-i_0) > 1 - \frac{\exp(-n-1)}{M}$ return a rational r at most $\exp(-n-1)$ far from $f(1 - \frac{\exp(-n-1)}{M}, x)$. We have $|\overline{f}(t,x) - r| \leq |\overline{f}(t,x) - f(1 - \frac{\exp(-n-1)}{M}, x)| + |f(1 - \frac{\exp(-n-1)}{M}, x) - r| \leq \exp(-n-1) + \exp(-n-1) \leq \exp(-n)$.

Otherwise $t_{i_0} - \exp(-i_0) \leq 1 - \frac{\exp(-n-1)}{M}$ implies that $t < t_{i_0} + \exp(-i_0) < 1$. Return a rational r at most $\exp(-n)$ far from $f(t,x)$.