An Analog Characterization of Elementarily Computable Functions over the Real Numbers

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Discrete, Continuous

- Discrete world: computing over $\mathbb{N}$ in discrete time.
  (Turing machines, automata...)
- Continuous world: computing over $\mathbb{R}$
  - in discrete time.
    (Recursive analysis, BSS machines)
  - in continuous time.
    (General Purpose Analog Computer, Neural networks...)

Discrete World

Church's thesis: All reasonable discrete computational models compute the same functions.

Turing machines, as well as 2-stack automata compute recursive functions ($\mathcal{R}ec = [0, S, U; COMP, REC, MU]$).

Sub-recursive functions

$$\mathcal{E} = [0, S, U, +, \ominus; COMP, BSUM, BPROD]$$

$$\mathcal{E}_n = [0, S, U, +, \ominus, E_{n-1}; COMP, BSUM, BPROD]$$

$$\mathcal{PR} = [0, U, S; COMP, REC]$$

With

$E_0(x, y) = x + y$

$E_1(x, y) = (x + 1) \times (y + 1)$

$E_2(x) = 2^x$

$E_{n+1}(x) = E_n^{[x]}(1)$ for $n \geq 2$ with $f^{[0]}(x) = x$, $f^{[d+1]}(x) = f(f^{[d]}(x))$

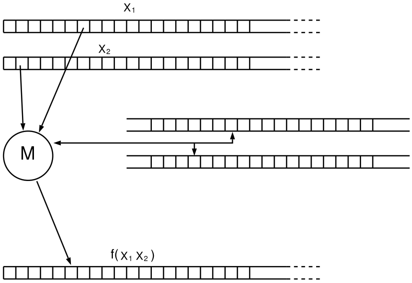
Continuous World

Several models:

- Recursive analysis
- GPAC
- $\mathbb{R}$-recursive functions
- Optical models
- ...

But no equivalent of Church thesis.

Recursive analysis: type 2 machines



A tape represents a real number:

Let $\nu_{\mathbb{Q}}$ be a representation of the rational numbers.
$(x_n) \rightsquigarrow x$ iff $\forall i, |x - \nu_{\mathbb{Q}}(x_i)| < \frac{1}{2^i}$

M behaves like a Turing Machine

Read-only one-way input tapes
Write-only one-way output tape.

Elementarily computable functions

Definition (Elementarily computable functions on compact domains)

A function $f : [a, b] \to \mathbb{R}$ with $a, b \in \mathbb{Q}$ is elementarily computable iff there exists $\phi : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ elementary such that

$$\forall X \rightsquigarrow x, (\phi(X)) \rightsquigarrow f(x).$$

Definition (Elementarily computable functions on other domains)

A function $f : [a, b) \to \mathbb{R}$ with $a, b \in \mathbb{Q}$ is elementarily computable iff there exists $\phi : \mathbb{N}^{\mathbb{N}} \times \mathbb{N} \to \mathbb{N}^{\mathbb{N}}$ elementary such that

$$\forall M < b, \forall x \in [a, M], \forall X \rightsquigarrow x, (\phi(X, M)) \rightsquigarrow f(x).$$

Class $\mathcal{G}$ ([Moore96])

$\mathcal{R}ec$	$\mathcal{G}$
0	0
S	1
U	U
$Comp$	$Comp$
$REC : f, g \mapsto h$ $h(x,0) = f(x)$ $h(x, n+1) = g(x, n, h(x,n))$	$INT : f, g \mapsto h$ $h(x,0) = f(x)$ $\partial_y h(x,y) = g(x,y,h(x,y))$
$MU : x, f \mapsto \min\{y; f(x,y) = 0\}$	$MU : x, f \mapsto \inf\{y \mid f(x,y) = 0\}$

Troubles with $\mathcal{G}$

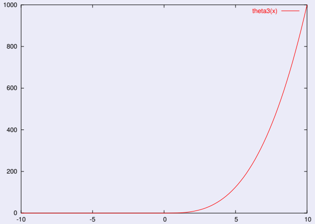
- Not always well defined ($0 \times +\infty = 0$, integration on non integrable functions...)
- Contains bad functions ($\chi_{\mathbb{Q}}$ which is nowhere continuous)
- Not physically reasonable (Zeno paradox $\Rightarrow$ infinite energy required)

Class $\mathcal{L}$ ([Campagnolo00])

$\mathcal{G}$	$\mathcal{L}'$
0	0
1	$1,\ -1,\ \pi$
U	U
$Comp$	$Comp$
$INT : f, g \mapsto h$ $h(x,0) = f(x)$ $\partial_y h(x,y) = g(x,y,h(x,y))$	$LI : f, g \mapsto h$ $h(x,0) = f(x)$ $\partial_y h(x,y) = g(x,y)h(x,y)$
MU	

Definition of $\mathcal{L}$

Definition (θ_3)



$$\theta_3(x) = \begin{cases} 0 & \text{if } x < 0 \\ x^3 & \text{if } x \geq 0 \end{cases}$$

Definition ($\mathcal{L}$)

$$\mathcal{L} = [0, 1, -1, \pi, U, \theta_3; COMP, LI]$$

Properties of $\mathcal{L}$

Theorem (Campagnolo)

$$\mathcal{L} \subset \mathcal{E}(\mathbb{R})$$

Theorem (Campagnolo)

$$\mathcal{E} \subset DP(\mathcal{L})$$

All discrete elementary functions have a real extension in $\mathcal{L}$.

Definition of $\mathcal{L}_n$

Definition ($\bar{E}_n$)

$\exp_2(n) = 2^n$
$\exp_{i+1}(n) = \exp_i^{[n]}(1)$
$\bar{E}_n$ is a monotonous real extension of $\exp_n$.

Definition ($\mathcal{L}_n$)

$$\mathcal{L}_n = [0, 1, -1, \pi, U, \theta_3, \bar{E}_{n-1}; COMP, LI].$$

Properties of $\mathcal{L}_n$

Theorem (Campagnolo)

$$\mathcal{L}_n \subset \mathcal{E}_n(\mathbb{R})$$

Theorem (Campagnolo)

$$\mathcal{E}_n \subset DP(\mathcal{L}_n)$$

All $\mathcal{E}_n$-computable functions over $\mathbb{N}$ have a real extension in $\mathcal{L}_n$.

Observation

$\mathcal{L}$ fails to characterize elementarily computable functions over the reals.

Question: How can we characterize elementarily computable functions over the reals?

Observation

$\mathcal{E}(\mathbb{R})$ is not stable by composition.

Definition of a weaker Composition schema

Definition (COMP)

$COMP(f,g)$ is defined only if there exists a product of closed intervals C such that $Range(f) \subseteq C \subset Domain(g)$.

$$COMP(f,g)(\overrightarrow{x}) = g(f(\overrightarrow{x})).$$

Remark: For total functions, this schema remains the classical one.

Definition of a limit operator

Definition (LIM)

Let $f : \mathbb{R} \times \mathcal{D} \to \mathbb{R}$ and a polynomial $\beta : \mathcal{D} \to \mathbb{R}$ such that $\exists K$ such that $\forall t, x$,
$\|\frac{\partial f}{\partial t}(t,x)\| \leq K \exp(-t\beta(x))$
$\|\frac{\partial^2 f}{\partial t \partial x}(t,x)\| \leq K \exp(-t\beta(x))$
Then, on an interval $I \subset \mathbb{R}$ where $\beta(x) > 0$, $F = LIM(f, \beta)$ is defined by

$$F(x) = \lim_{t \to +\infty} f(t,x)$$

if this function is $\mathcal{C}^2$.

New classes

$$\mathcal{L}^* = [0, 1, -1, U, \theta_3; COMP, LI, LIM]$$

$$\mathcal{L}^*_n = [0, 1, -1, U, \theta_3, \bar{E}_{n-1}; COMP, LI, LIM]$$

Basic properties of $\mathcal{L}^*$

- $\frac{1}{x} : \left\{ \begin{array}{ccc} \mathbb{R}^{>0} & \to & \mathbb{R} \\ x & \mapsto & \frac{1}{x} \end{array} \right.$ belongs to $\mathcal{L}^*$:

  Let $E = LI(0, \exp(-tx))$. $E(t,x) = \left\{ \begin{array}{ll} \frac{1-\exp(-tx))}{x} & \text{if } x \neq 0 \\ t & \text{if } x = 0 \end{array} \right.$.

  And $\frac{1}{x} = LIM(E, x \mapsto x)$.
- $\pi \in \mathcal{L}^*$:

  $1 + x^2 \in \mathcal{L}^*$, $\frac{1}{1+x^2} \in \mathcal{L}^*$.

  arctan $= LI(0, \frac{1}{1+x^2})$ and $\pi = 4 \arctan(1)$.
- $\mathcal{L} \subsetneq \mathcal{L}^*$

Characterization of $\mathcal{E}(\mathbb{R})$

Proposition

$$\mathcal{L}^* \subseteq \mathcal{E}(\mathbb{R})$$

Proposition

Let f a $\mathcal{C}^2$ function defined on a compact domain,
if $f \in \mathcal{E}(\mathbb{R})$, then $f \in \mathcal{L}^*$.

Characterization of $\mathcal{E}(\mathbb{R})$

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Proposition

Let f a $\mathcal{C}^2$ function defined on a compact domain,
if $f \in \mathcal{E}(\mathbb{R})$, then $f \in \mathcal{L}^*$.

Theorem

If f is of class $\mathcal{C}^2$, has a compact domain,

$$f \in \mathcal{E}(\mathbb{R}) \Leftrightarrow f \in \mathcal{L}^*.$$

Characterization of $\mathcal{E}_n(\mathbb{R})$

Proposition

$$\mathcal{L}_n^* \subseteq \mathcal{E}_n(\mathbb{R})$$

Proposition

Let f a $\mathcal{C}^2$ function defined on a compact domain,
if $f \in \mathcal{E}_n(\mathbb{R})$, then $f \in \mathcal{L}_n^*$.

Theorem

If f is of class $\mathcal{C}^2$, has a compact domain,

$$f \in \mathcal{E}_n(\mathbb{R}) \Leftrightarrow f \in \mathcal{L}_n^*$$

Results

- Machine-independent characterization of real elementarily computable functions.
- For $\mathcal{C}^2$ functions defined on a compact domain,

$$f \in \mathcal{E}(\mathbb{R}) \Leftrightarrow f \in \mathcal{L}^*$$

$$f \in \mathcal{E}_n(\mathbb{R}) \Leftrightarrow f \in \mathcal{L}^*_n$$

- Normal form:
  If $f \in \mathcal{L}^*$ or $\mathcal{L}^*_n$, f can be written with at most two LIM.

Perspectives

- What about functions over $\mathbb{R}$?
- Improving our characterization:
  - weaker limit schema
  - avoiding limits
  - improving normal form theorem
- Characterizing computable functions over the reals