Linearization by Program Transformation*

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Motivation

- Linear terms have good properties:
  - ⋆ Strongly normalizable
  - ⋆ Non duplicating reduction
  - ⋆ Typable in polynomial time
- Existing work in the area:
  - ⋆ (Kfoury) Indirect relation between Λ and a (new) linear calculus
  - ⋆ (Damas and Florido) Relation between linear terms and terms typable by intersections

Introduction

- Restricted class of λ-terms

$$\text{Weak Linear Lambda Calculus } (\Lambda_{\mathcal{WL}})$$

- Transformation

$$M \qquad\qquad\qquad\qquad M \in \Lambda$$

Introduction

- Restricted class of λ-terms

Weak Linear Lambda Calculus ($\Lambda_{\mathcal{WL}}$)

- Transformation

$$\begin{array}{ccc} M & & M \in \Lambda \\ \downarrow \mathcal{T} & & \\ \mathcal{T}(M) & & \mathcal{T}(M) \in \Lambda_{\mathcal{WL}} \end{array}$$

Plan

- Weak Linear Lambda Calculus
- Type System
- Transformation
  - ⋆ Labeled Lambda Calculus
  - ⋆ Paths
  - ⋆ Transformation
- Conclusions

Weak Linear Lambda Calculus

Weak Linear Terms

A λ-term M is weak linear if all the β-redexes in any reduction sequence starting from M are non duplicating

Weak Linear Terms

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Example:

$(\lambda x_1 x_2 . x_1 x_2)(\lambda x.x)(\lambda x.x) \to_\beta^*$
$(\lambda x.x)(\lambda x.x) \to_\beta$
$(\lambda x.x)$

$(\lambda x.xx)(\lambda x.x) \to_\beta$
$(\lambda x.x)(\lambda x.x) \to_\beta$
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Weak Linear Terms

A λ-term M is weak linear if all the β-redexes in any reduction sequence starting from M are non duplicating

Example:

$(\lambda x_1 x_2.x_1 x_2)(\lambda x.x)(\lambda x.x) \rightarrow^*_\beta$ $(\lambda x.x)(\lambda x.x) \rightarrow_\beta$ $(\lambda x.x)$	$(\lambda x.xx)(\lambda x.x) \rightarrow_\beta$ $(\lambda x.x)(\lambda x.x) \rightarrow_\beta$ $(\lambda x.x)$

That is:

$(\lambda x_1 x_2.x_1 x_2)(\lambda x.x)(\lambda x.x)$ is weak linear	$(\lambda x.xx)(\lambda x.x)$ is not

Weak linear terms have nice properties

- Strong normalization

  1. Non-duplicating reduction
  2. Weak linear term are strongly normalizable

- It is decidable to know if a λ-term is Weak Linear

Type System

Type System - I

- Based on intersection types

$$\sigma ::= \alpha \mid \tau_1 \cap \cdots \cap \tau_n \to \sigma$$

- Use intersections to type abstractions
- Domain of applyed functions are not intersections

Type System - II

VAR	$\{x : \sigma\} \vdash x : \sigma$	
ABS-I	$\dfrac{A \cup \{x : \tau_1, \ldots, x : \tau_n\} \vdash M : \sigma}{A \vdash \lambda x.M : \tau_1 \cap \cdots \cap \tau_n \rightarrow \sigma}$	if $x \in FV(M)$
ABS-K	$\dfrac{A \vdash M : \sigma}{A \vdash \lambda x.M : \tau \rightarrow \sigma}$	if $x \notin FV(M)$
APP	$\dfrac{A_1 \vdash M : \tau \rightarrow \sigma \quad A_2 \vdash N : \tau}{A_1 \cup A_2 \vdash MN : \sigma}$	

Type Inference

- Type Inference Algorithm - $\mathcal{I}$
- Sound and Complete
- Polynomial
- If M is weak linear, then $\exists A, \sigma . A \vdash M : \sigma$

Transforming terms into weak linear

Example

$$(\lambda xy.xy)(\lambda z.zz)(\lambda x.x)$$

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$(\lambda xy.xy)(\lambda z.zz)(\lambda x.x) \rightarrow_\beta$
$(\lambda y.(\lambda z.zz)y)(\lambda x.x)$

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linear$(\lambda z.zz) = (\lambda z_1 z_2.z_1 z_2)$

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$(\lambda xy.xyy)(\lambda z_1 z_2.z_1 z_2)(\lambda x.x) \to_\beta$
$(\lambda y.(\lambda z_1 z_2.z_1 z_2)yy)(\lambda x.x)$

Example

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$(\lambda x.\lambda y.xyy)(\lambda z_1 z_2.z_1 z_2)(\lambda x.x)$

linear$(\lambda y.xyy) = (\lambda y_1 y_2.xy_1 y_2)$

Example

$$(\lambda xy.xy)(\lambda z.zz)(\lambda x.x)$$

$(\lambda xy.xy)(\lambda z.zz)(\lambda x.x)$

$(\lambda xy.xy)(\lambda z.zz)(\lambda x.x) \rightarrow_\beta$
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linear$(\lambda z.zz) = (\lambda z_1 z_2.z_1 z_2)$

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linear$(\lambda y.xyy) = (\lambda y_1 y_2.xy_1 y_2)$

$$(\lambda xy_1 y_2.xy_1 y_2)(\lambda z_1 z_2.z_1 z_2)(\lambda x.x)(\lambda x.x)$$

Labeled Lambda Calculus

Labels $\mathcal{L}$:

$$l_1, l_2 \in \mathcal{L} ::= a \mid l_1 l_2 \mid \overline{l_1} \mid \underline{l_2}$$

Labeled lambda terms $\Lambda_{\mathcal{V}}^{\mathcal{L}}$:

$$M, N \in \Lambda_{\mathcal{V}}^{\mathcal{L}} ::= x^l \mid (\lambda x.M)^l \mid (MN)^l$$

example ▷

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example ▷

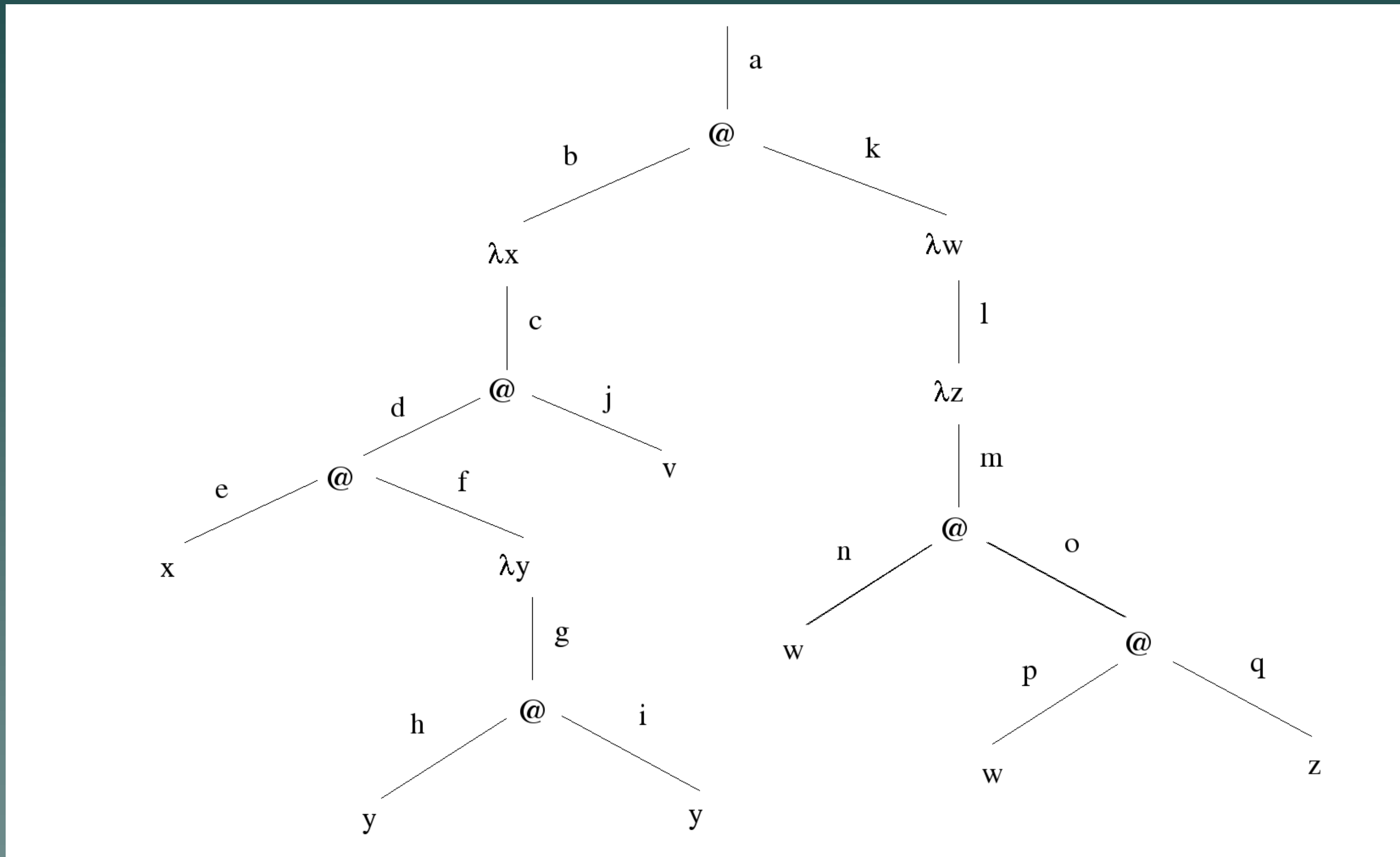
β-reduction:

degree of the redex

$$((\lambda x.M)^{l_0} N)^{l_1} \to l_1 \cdot \overline{l_0} \cdot M[\underline{l_0} \cdot N/x]$$

example ▷

Example



Example

Let $\Delta = (\lambda y.(y^h y^i)^g)^f$ and $P = (\lambda w.(\lambda z.(w^n (w^p z^q)^o)^m)^l)$ in

$$((\lambda x.((x^e \Delta)^d v^j)^c)^b P^k)^a$$

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$$((\lambda x.((x^e \Delta)^d v^j)^c)^b P^k)^a$$

One labeled reduction:

$$((\lambda x.((x^e \Delta)^d v^j)^c)^b P^k)^a \longrightarrow ((P^{e\,\underline{b}\,k} \Delta')^d z^j)^{a\,\overline{b}\,c}$$

Labels, Paths and Legal Paths

Different degrees correspond to different paths:

$$
\begin{aligned}
\text{path}(a) &= a \\
\text{path}(l_1 l_2) &= \text{path}(l_1) \cdot \text{path}(l_2) \\
\text{path}(\bar{l}) &= \text{path}(l) \\
\text{path}(\underline{l}) &= (\text{path}(l))^r
\end{aligned}
$$

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\end{aligned}
$$

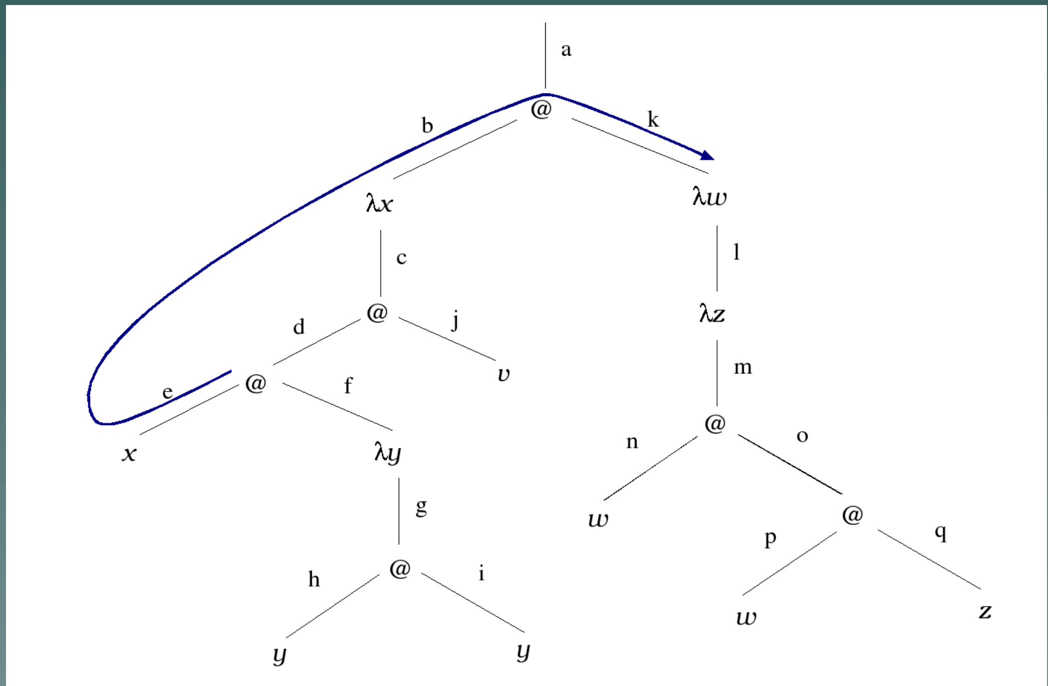
Legal paths characterize paths yield by degrees:

- Path yield by the degree of a redex $\Rightarrow$ legal path.
- Let φ be a legal path of type @-λ in M. Then $\exists l.M \rightarrow_{\beta*} (\lambda x.P)^l N$ and $path(l) = \varphi$.

Example

Let $\Delta' = (\lambda y.(y^h y^i)^g)^f$ and $P = (\lambda w.(\lambda z.(w^n(w^p z^q)^o)^m)^l)$.

degree$((P^{e\,\underline{b}\,k}\Delta')^d) = e\,\underline{b}\,k$, and path$(e\,\underline{b}\,k) = e\,b\,k$



One step of transformation

- $(l, k) = \text{next\_non\_linear\_path}(M)$
- n is the number of free occurrences of x in P (in $(\lambda x.P)^l$)

$$\mathcal{L}(M) = \text{expand}(n, M, l, k)$$

Example - I

Let

$$M = ((\lambda x.((x^e(\lambda y.(y^h y^i)^g)^f)^d v^j)^c)^b(\lambda w.(\lambda z.(w^n(w^p z^q)^o)^m)^l)^k)^a$$

The set of legal paths of type @-λ is

$$\{b : @\text{-}\lambda, e \cdot b \cdot k : @\text{-}\lambda, d \cdot e\, b\, k \cdot l : @\text{-}\lambda, n \cdot k\, b\, e \cdot f\ : @\text{-}\lambda, p \cdot k\, b\, e \cdot f\ : @\text{-}\lambda\}$$

Example - I

Let

$$M = ((\lambda x.((x^e(\lambda y.(y^h y^i)^g)^f)^d v^j)^c)^b(\lambda w.(\lambda z.(w^n(w^p z^q)^o)^m)^l)^k)^a$$

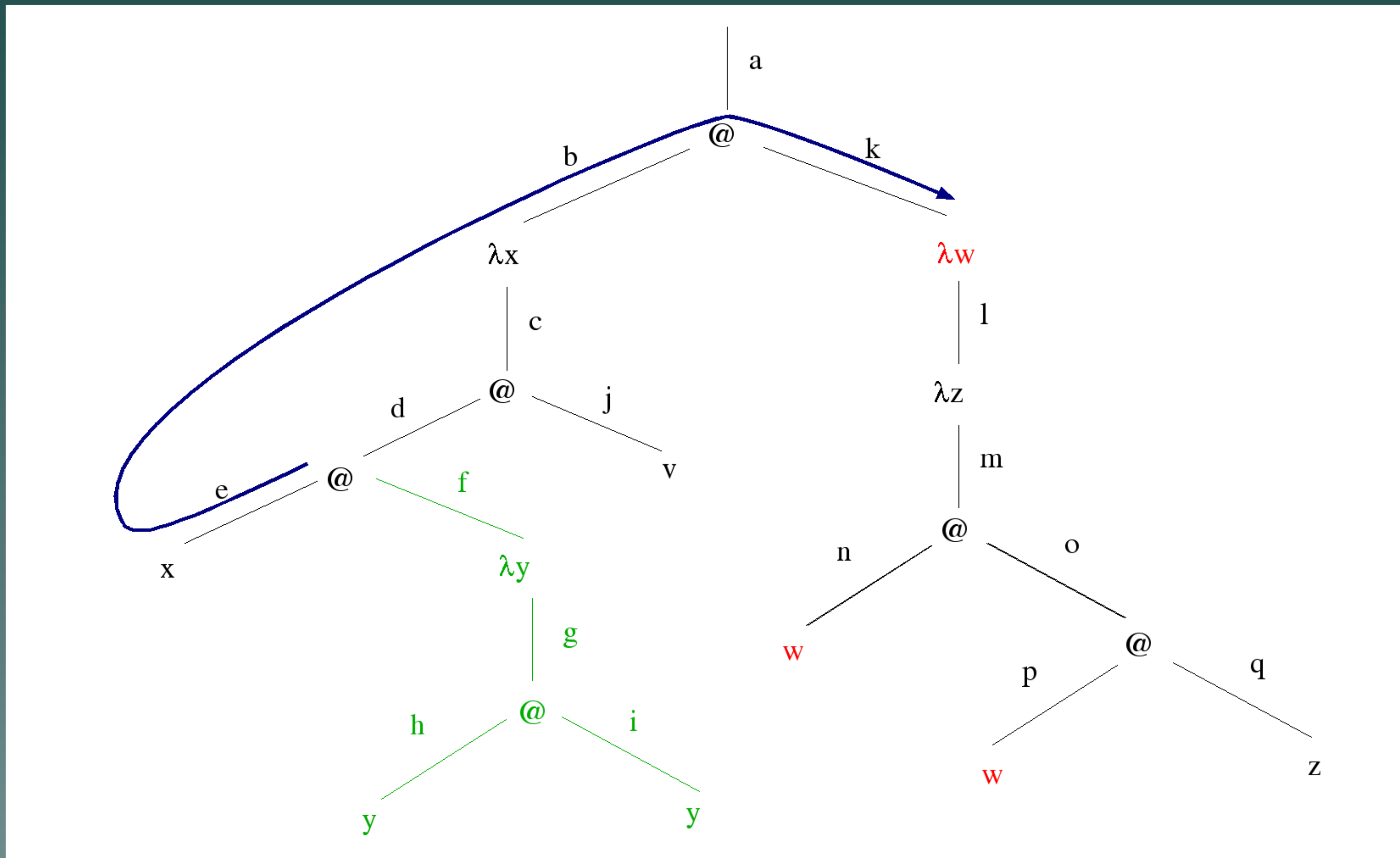
The set of legal paths of type @-λ is

$$\{b : @\text{-}\lambda, e \cdot b \cdot k : @\text{-}\lambda, d \cdot e\, b\, k \cdot l : @\text{-}\lambda, n \cdot k\, b\, e \cdot f : @\text{-}\lambda, p \cdot k\, b\, e \cdot f : @\text{-}\lambda\}$$

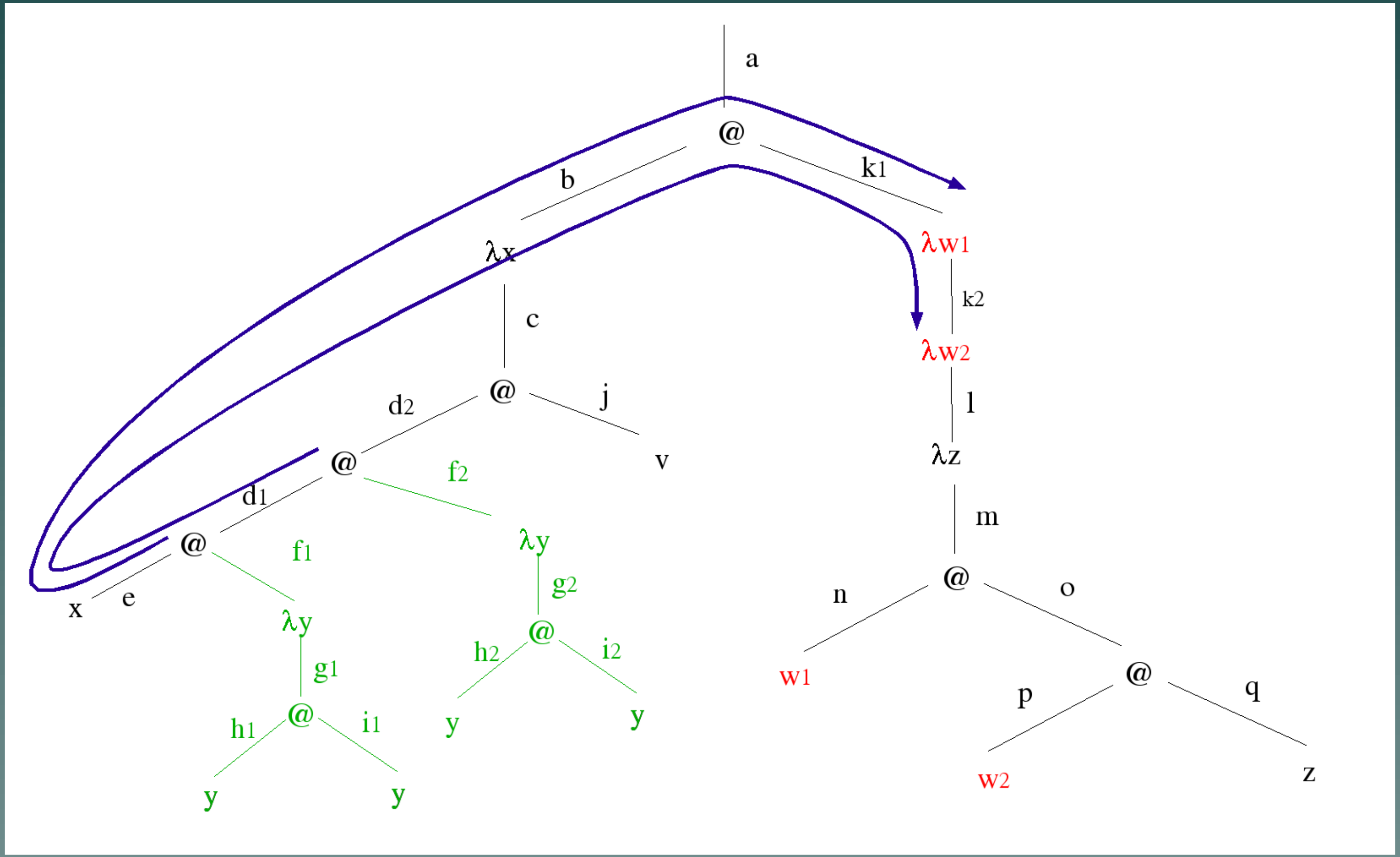
and $next\_non\_linear(M)$ gives

$$\{b : @\text{-}\lambda, e \cdot b \cdot k : @\text{-}\lambda, d \cdot e\, b\, k \cdot l : @\text{-}\lambda, n \cdot k\, b\, e \cdot f : @\text{-}\lambda, p \cdot k\, b\, e \cdot f : @\text{-}\lambda\}$$

expand(2,M,k,e)



expand(2,M,k,e)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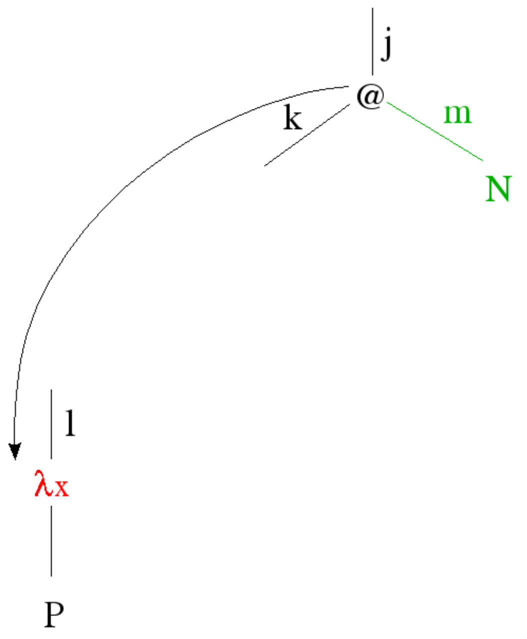
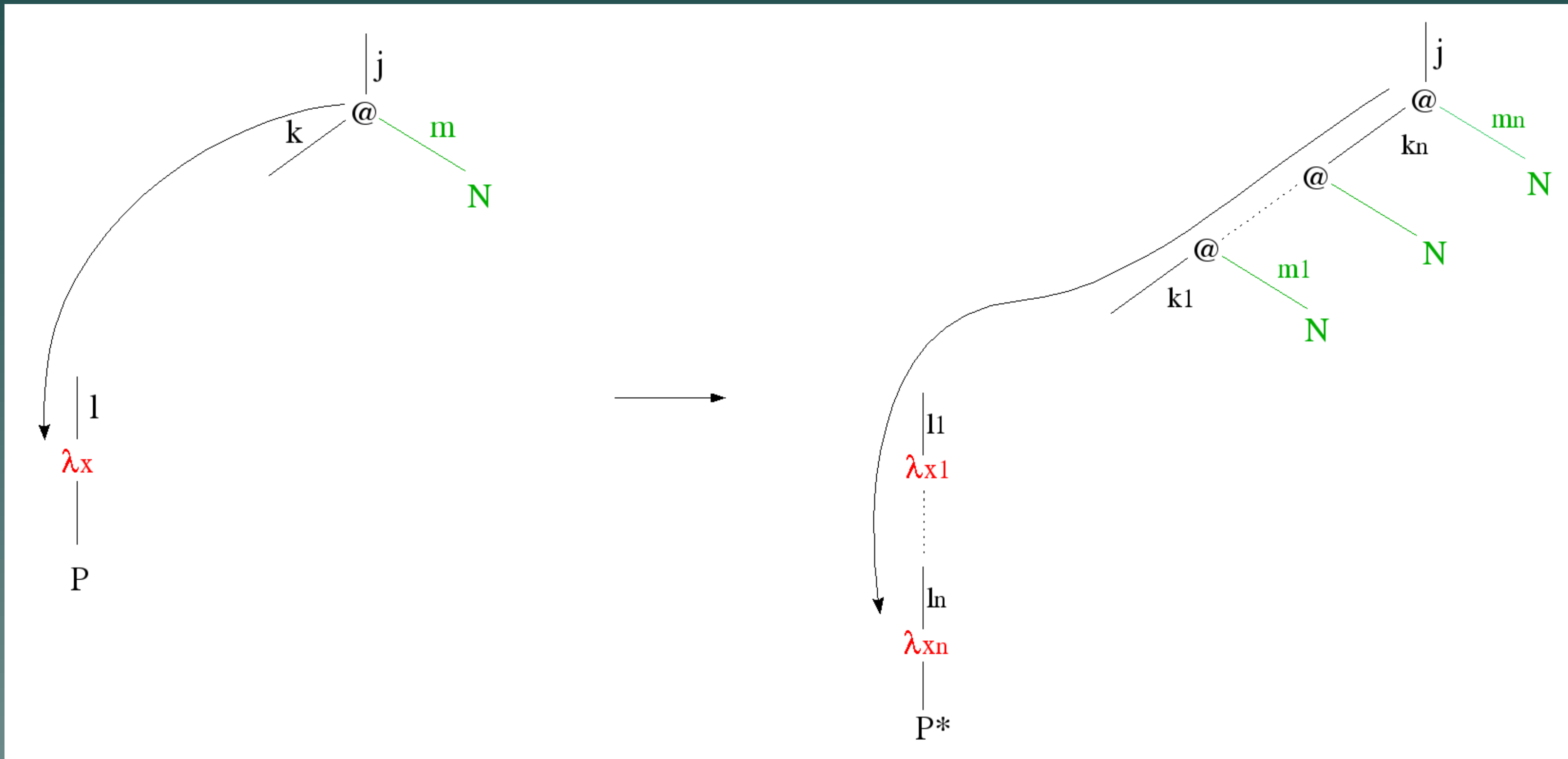
expand(n,M,l,k)



expand(n,M,l,k)



Transformation

$$\mathcal{T}(M) = \begin{cases} M & \text{if all\_linear}(\mathcal{LP}) \\ \mathcal{T}(\mathcal{L}(M)) & \text{otherwise} \end{cases}$$

Transformation

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The $\mathcal{T}$ transformation properties:

- $\mathcal{T}$ preserves normal forms
- $\mathcal{T}$ transforms terms into weak linear terms

Example

$$\mathcal{T}(\lambda x.x(\lambda y.yy)v)(\lambda fz.f(fz))$$

Notation:

- $\Delta = (\lambda y.yy)$
- $D = (\lambda y_1 y_2.y_1 y_2)$

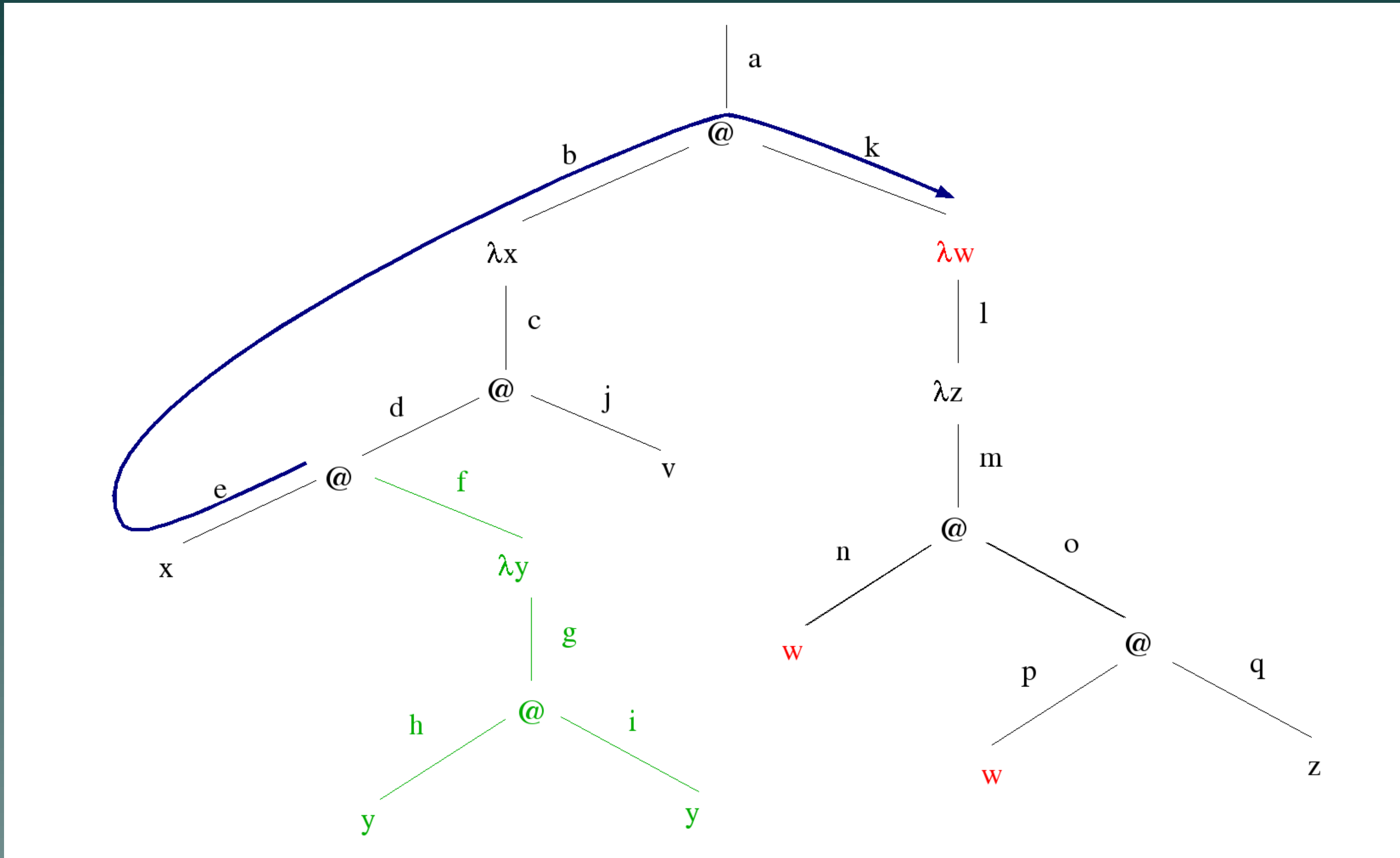
Example

$$\mathcal{T}(\lambda x.x(\lambda y.yy)v)(\lambda fz.f(fz))$$

Notation:

- $\Delta = (\lambda y.yy)$
- $D = (\lambda y_1 y_2.y_1 y_2)$

$(\lambda x.x(\lambda y.yy)v)(\lambda fz.f(fz)) \rightarrow_\beta (\lambda fz.f(fz))\Delta v \rightarrow_\beta \Delta(\Delta v) \rightarrow_\beta$
$(\Delta v)(\Delta v) \rightarrow_\beta (vv)(vv)$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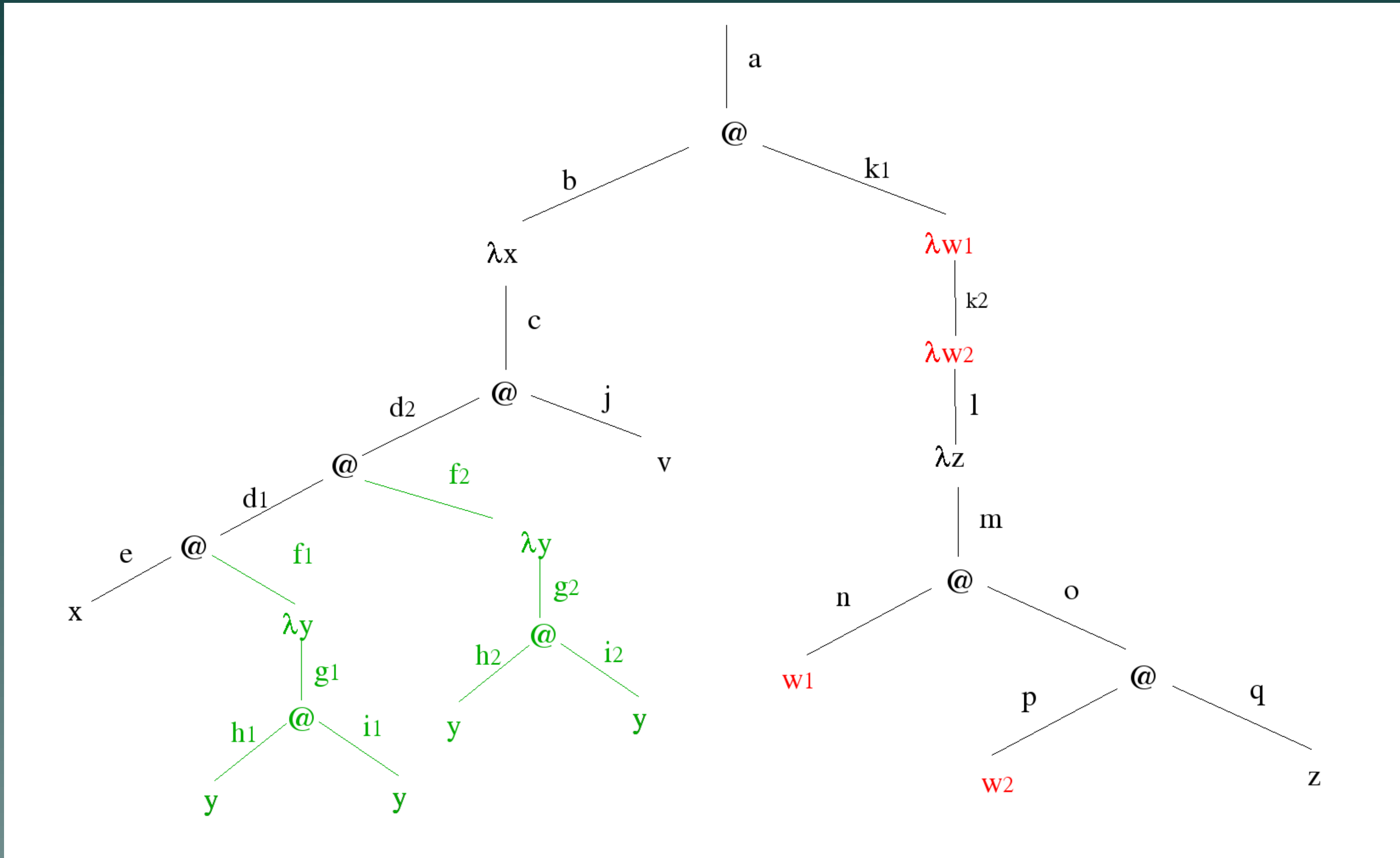
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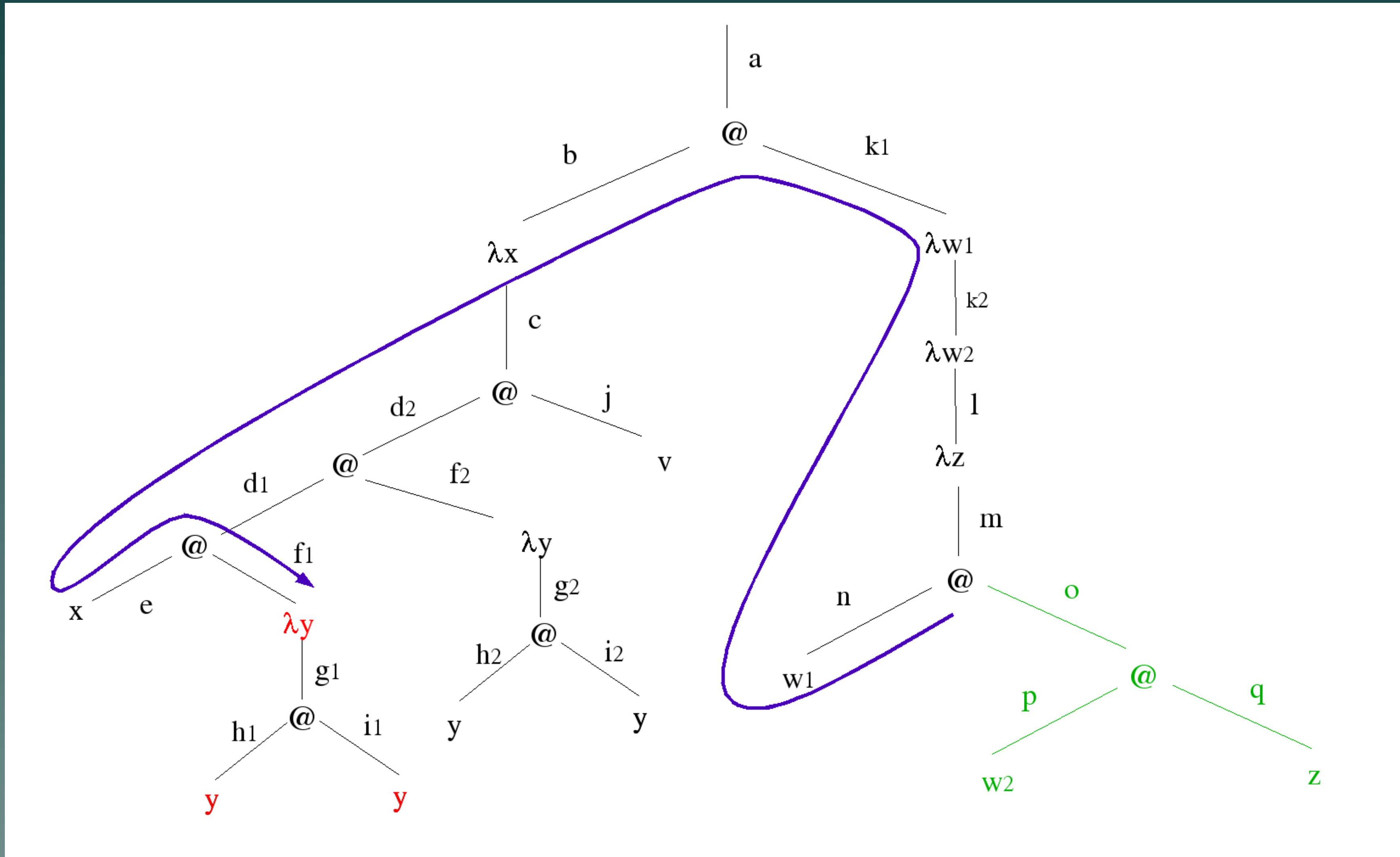
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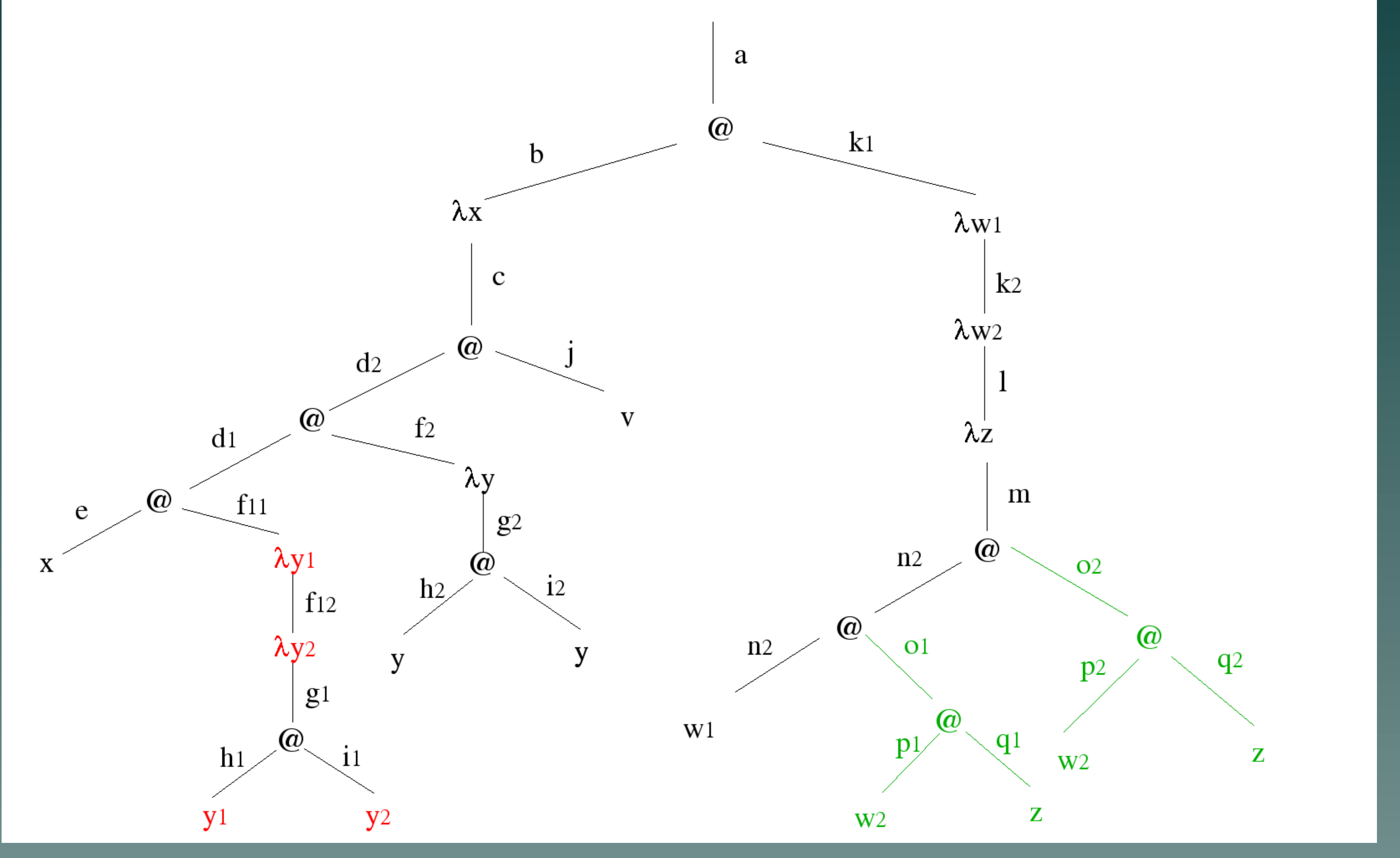
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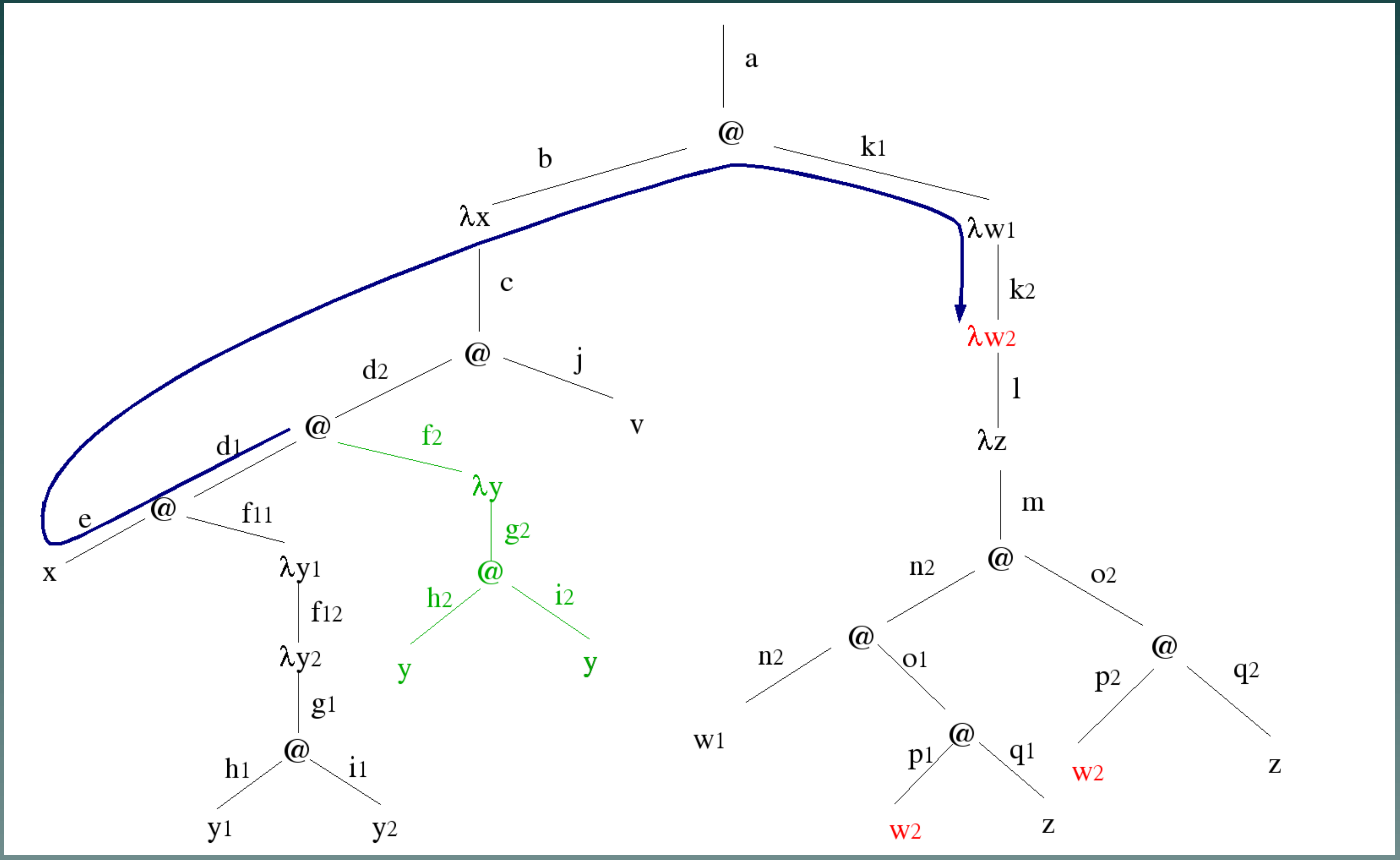
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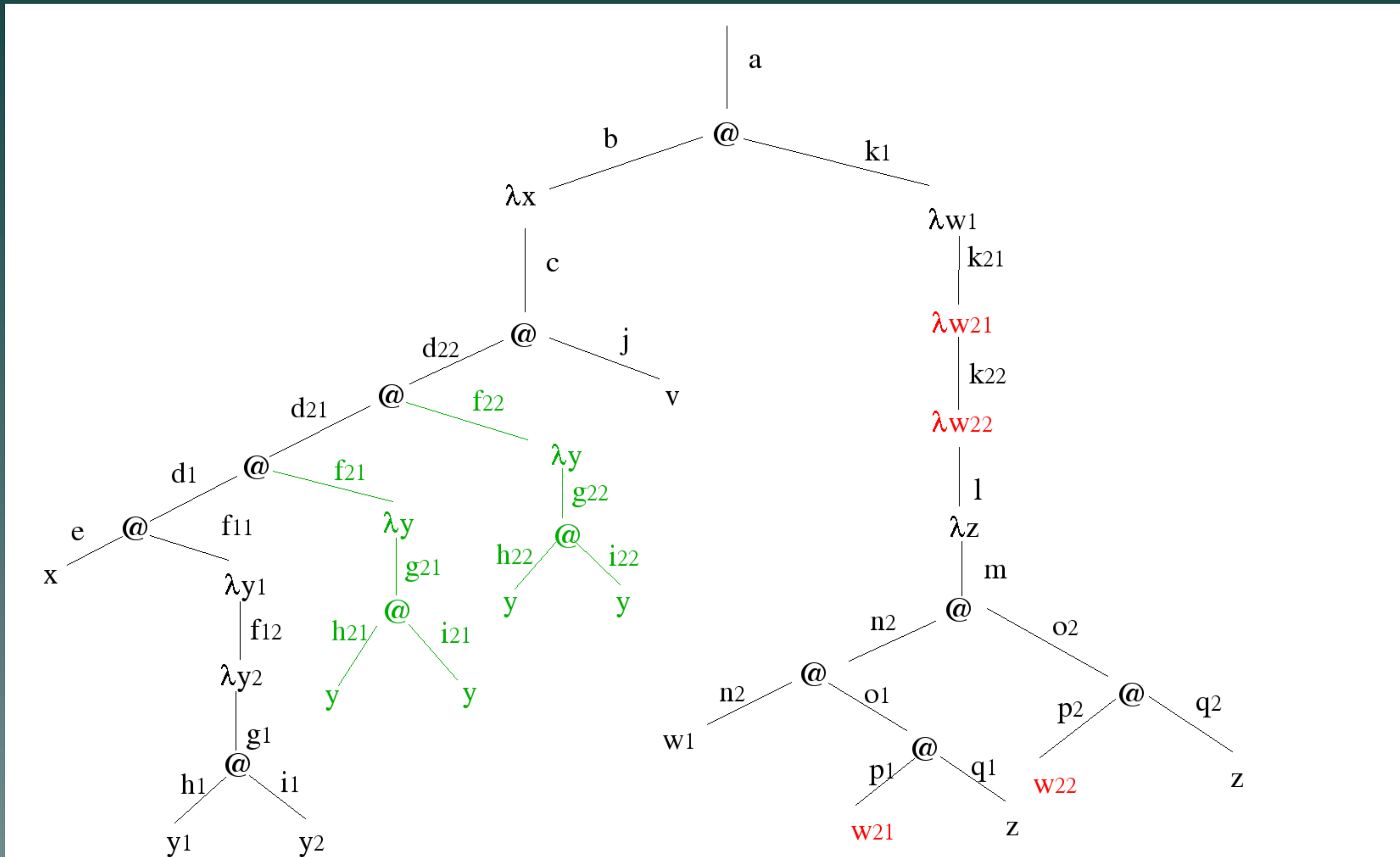
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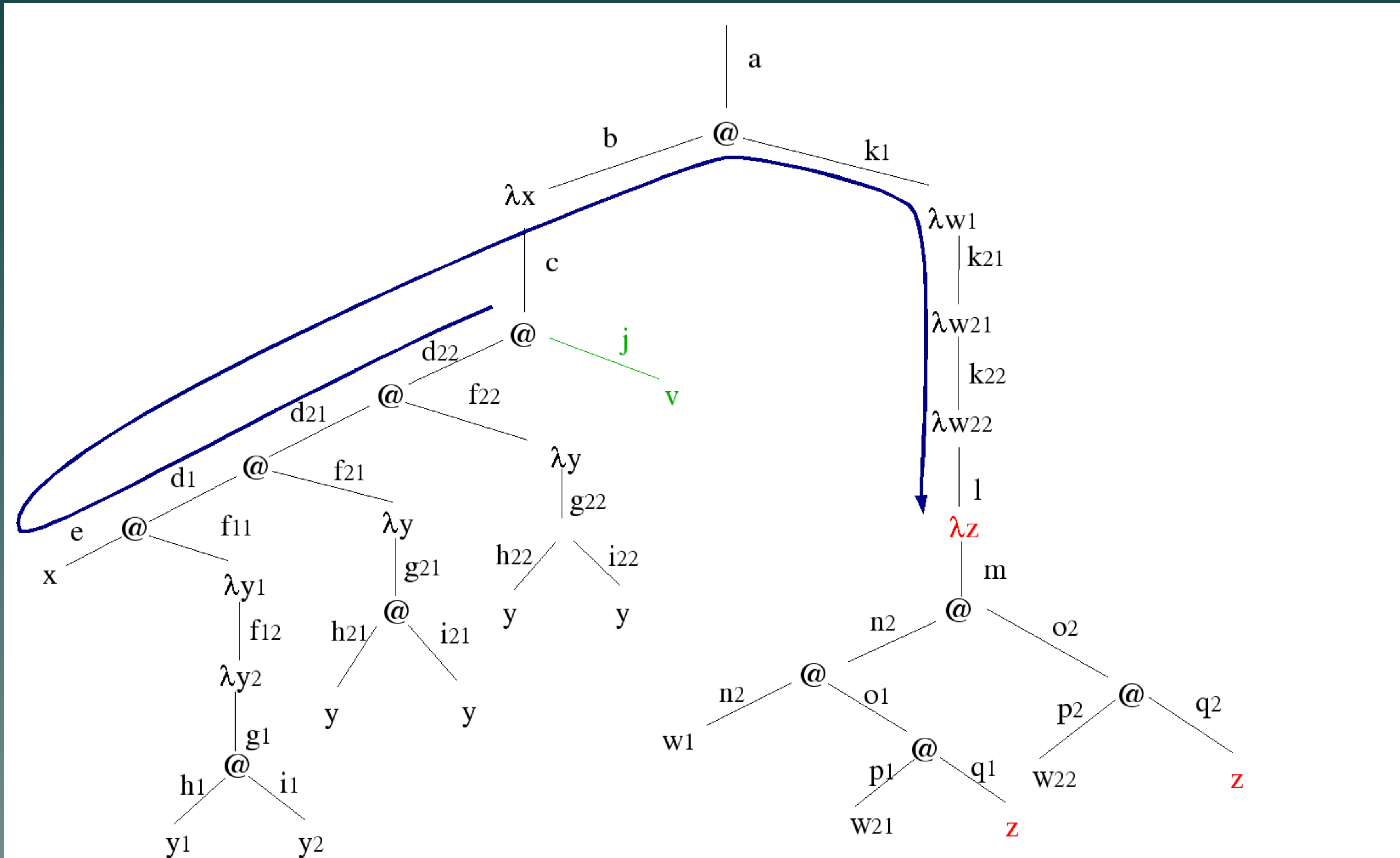
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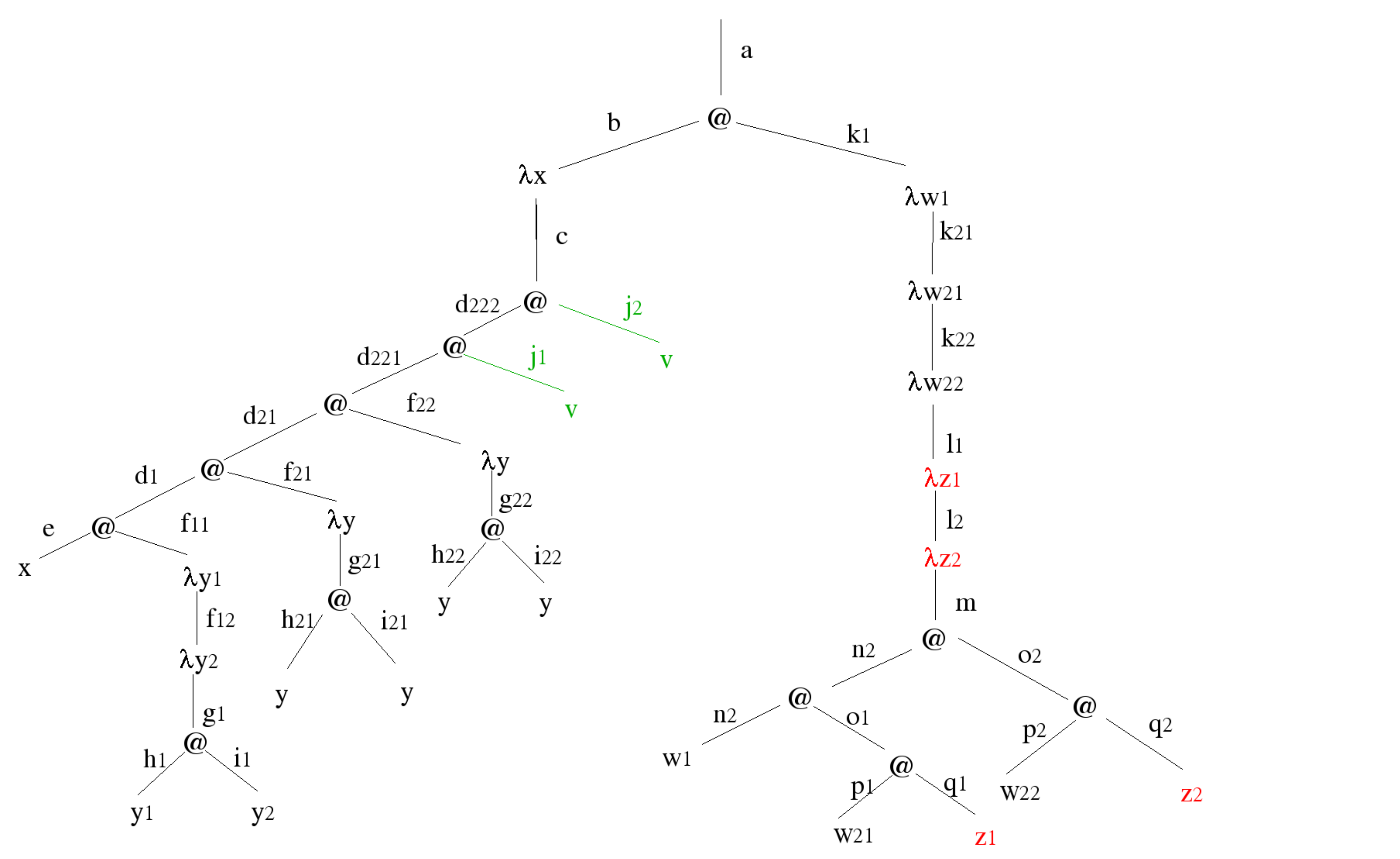
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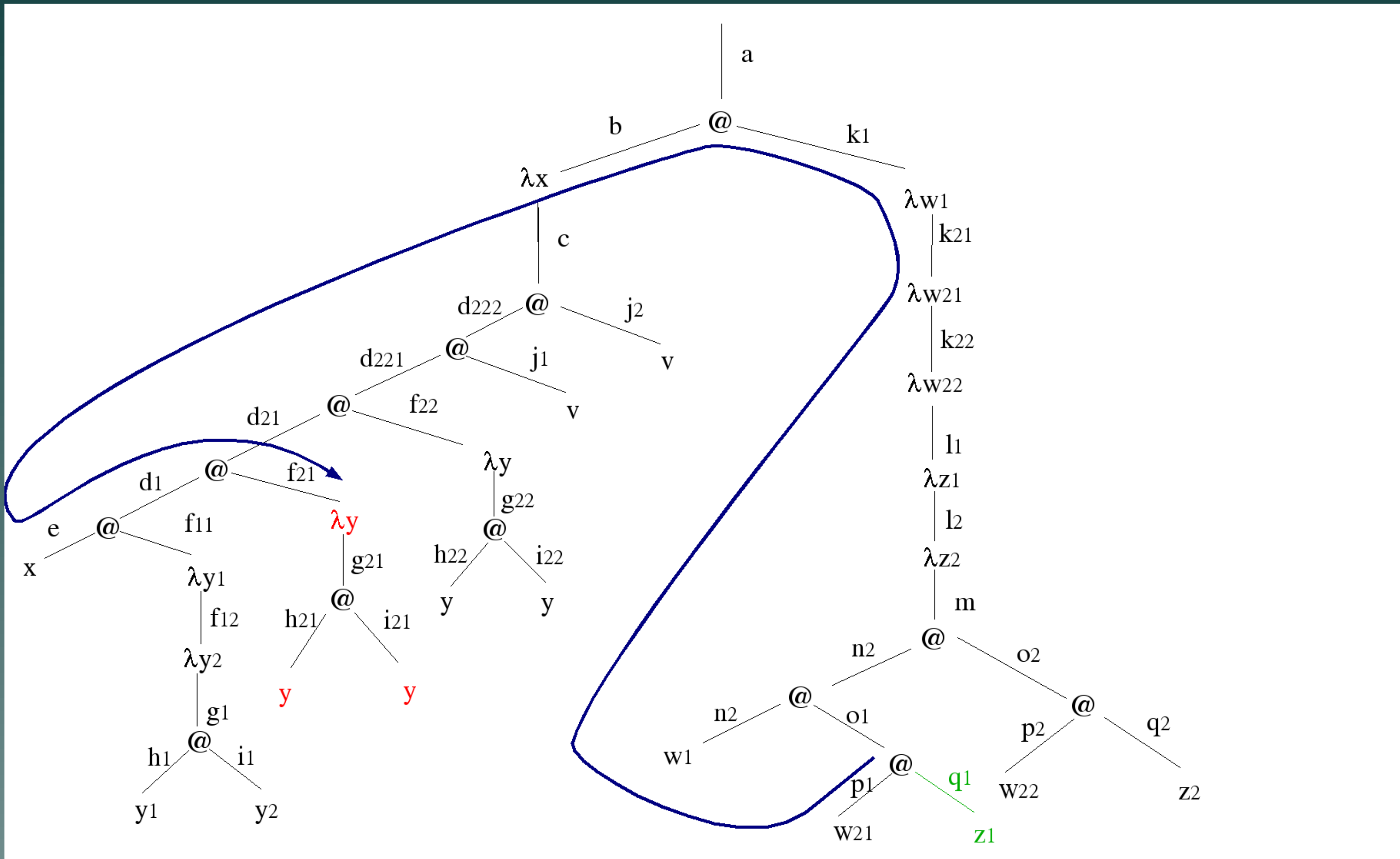
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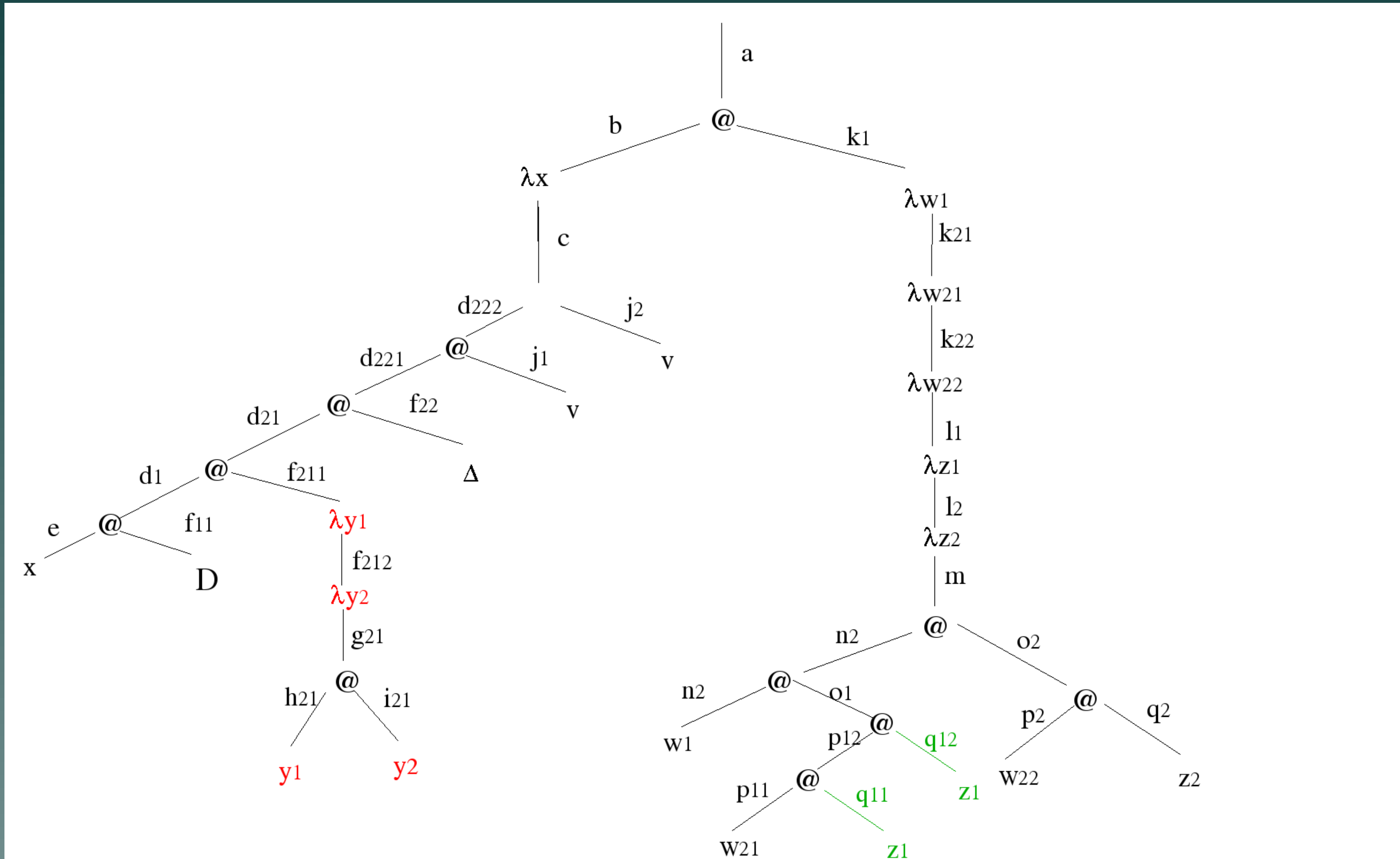
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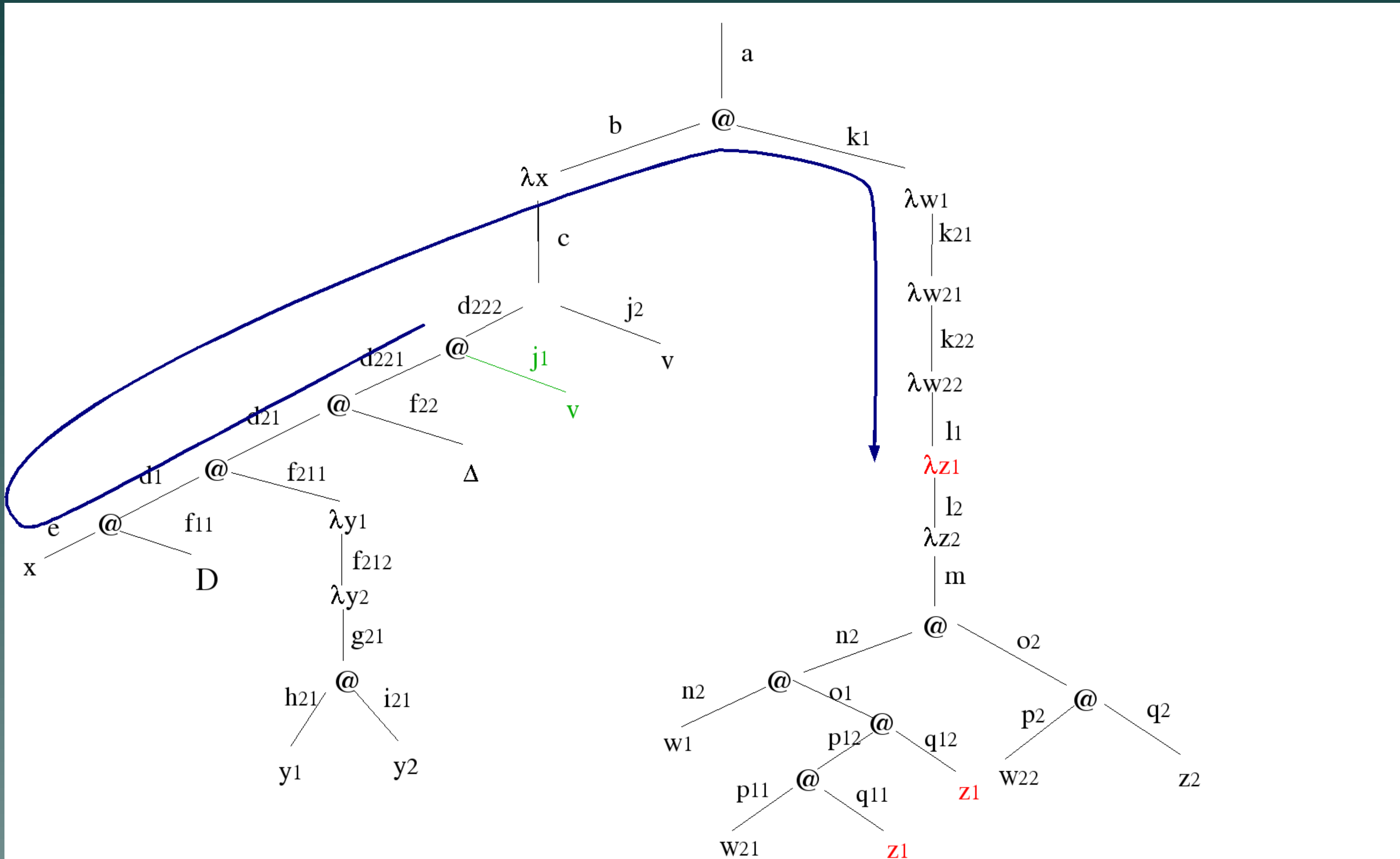
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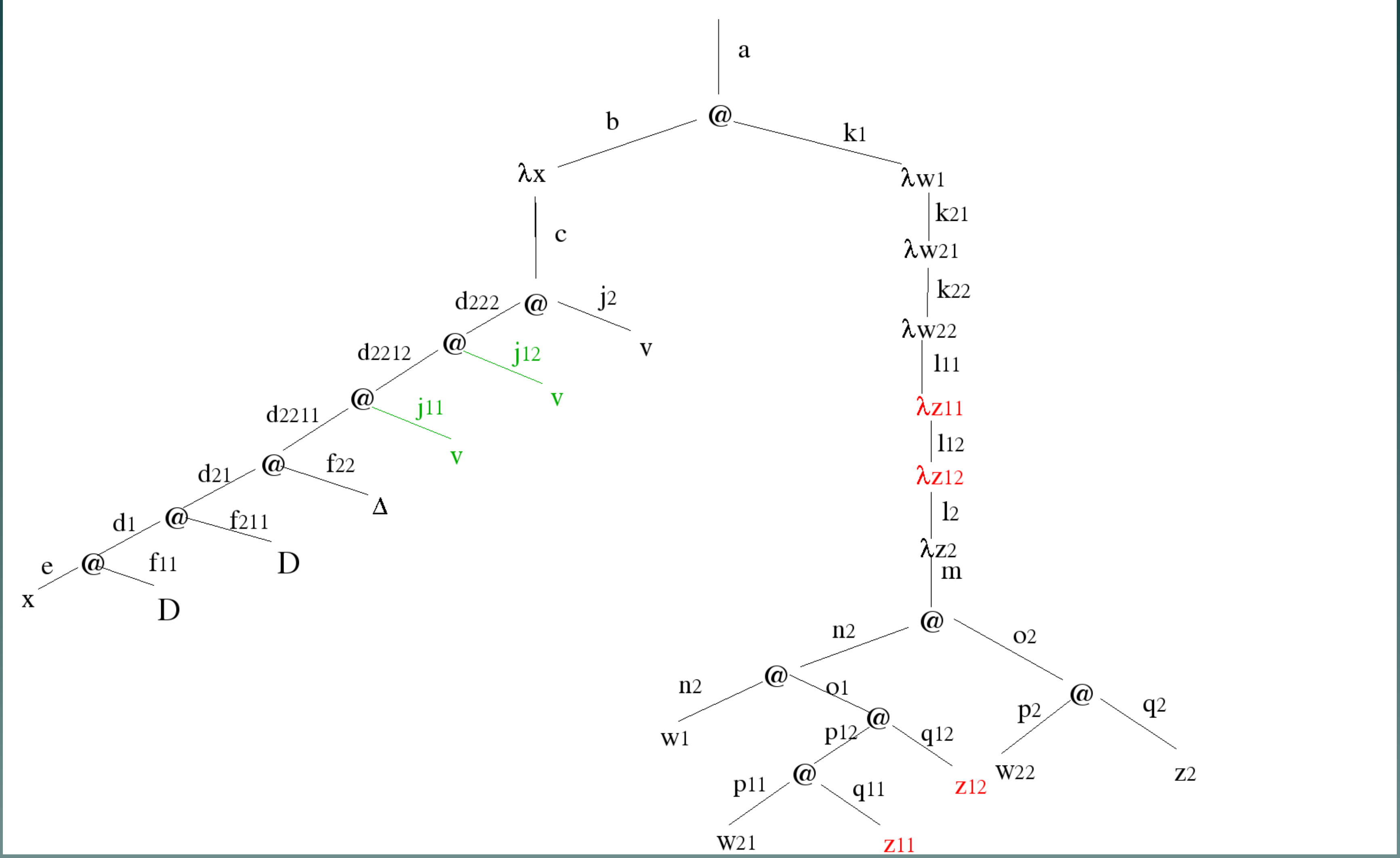
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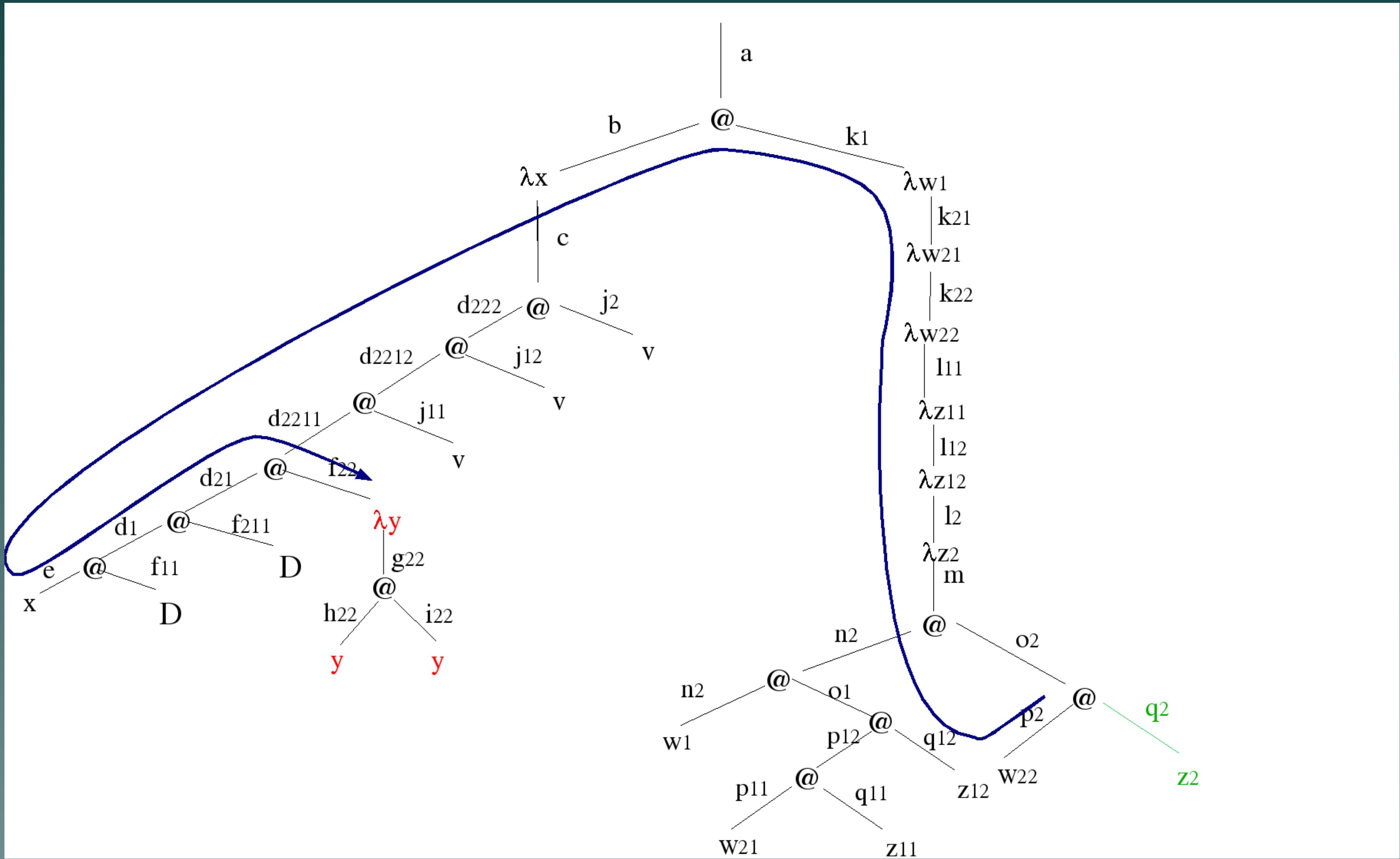
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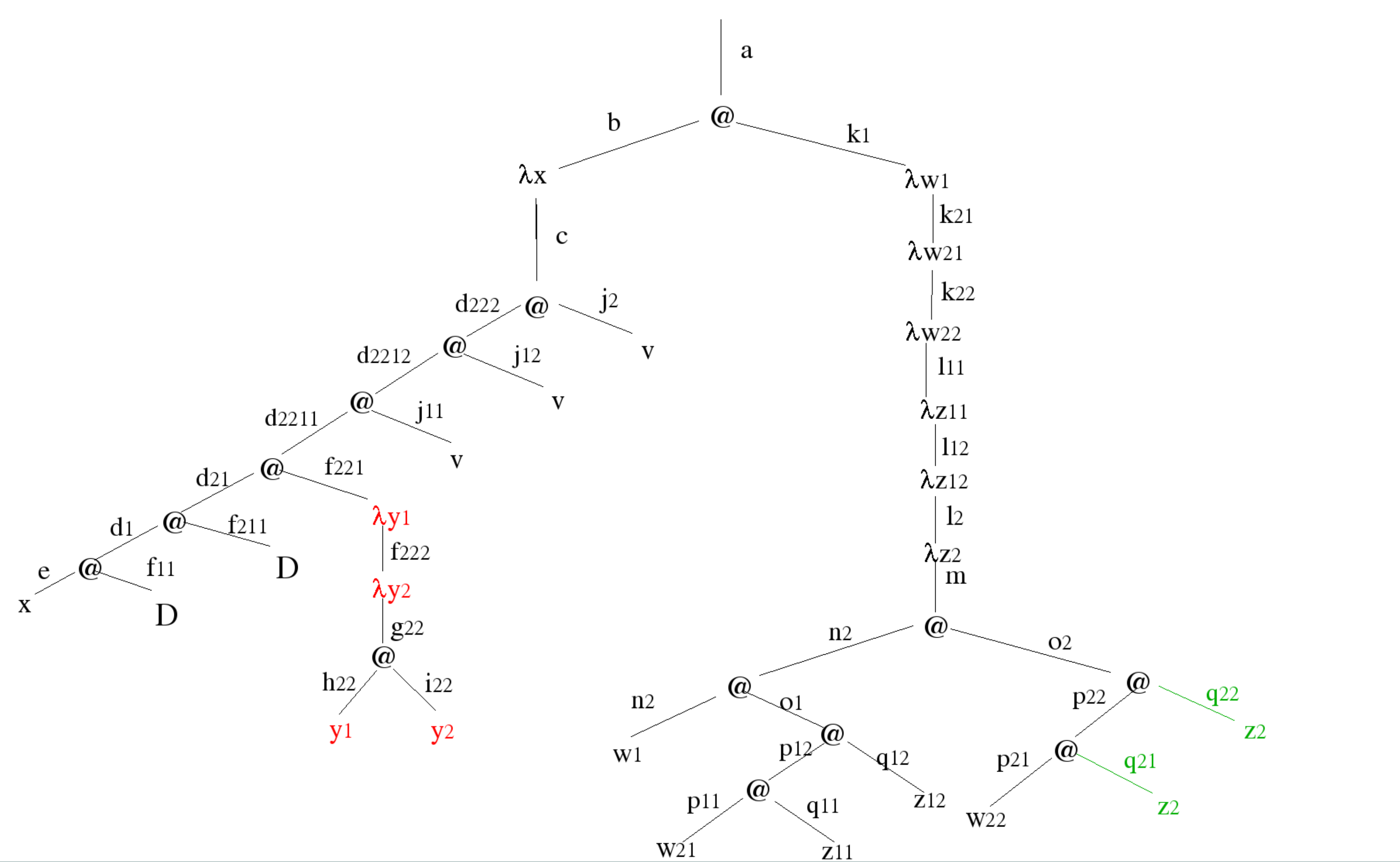
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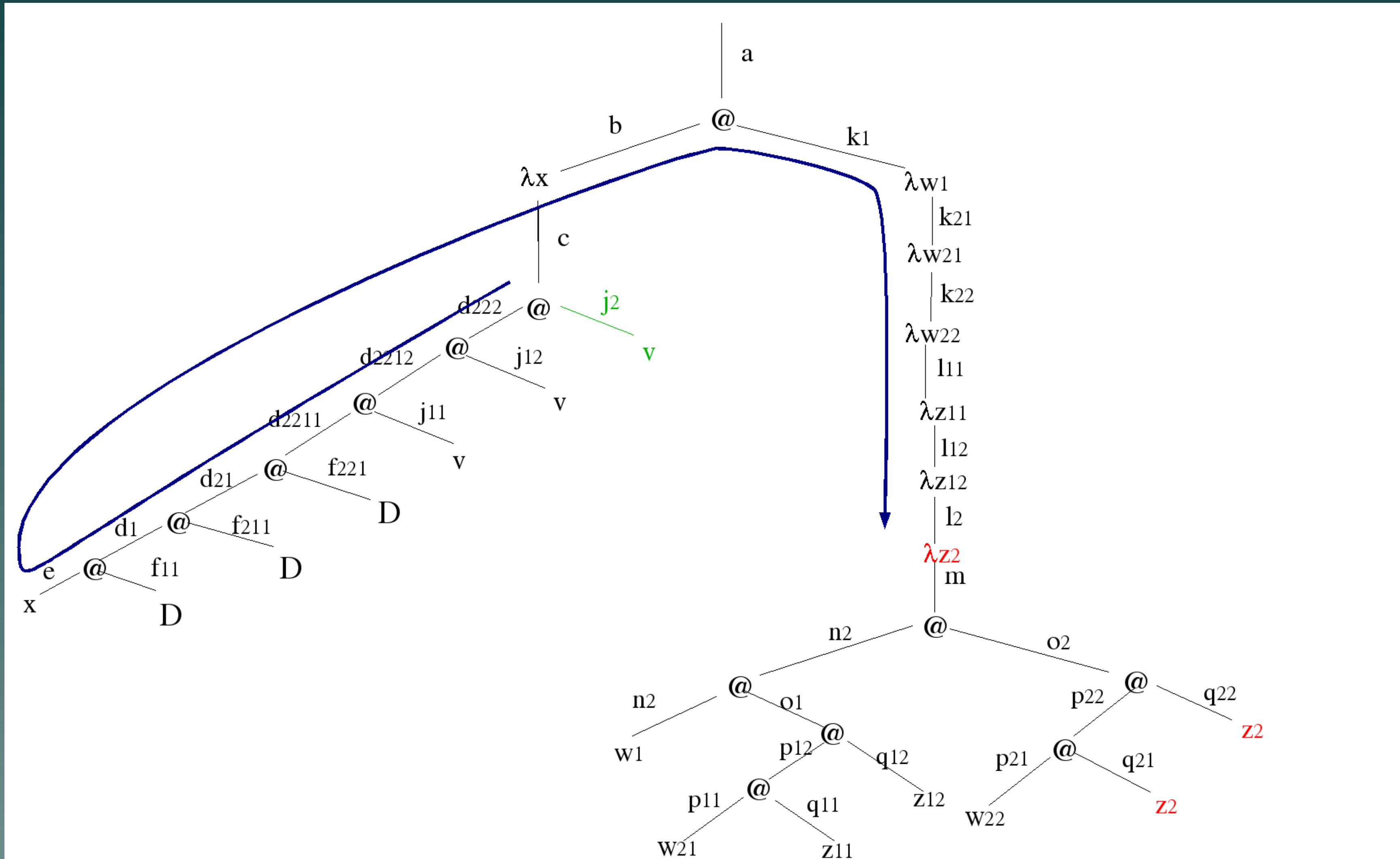

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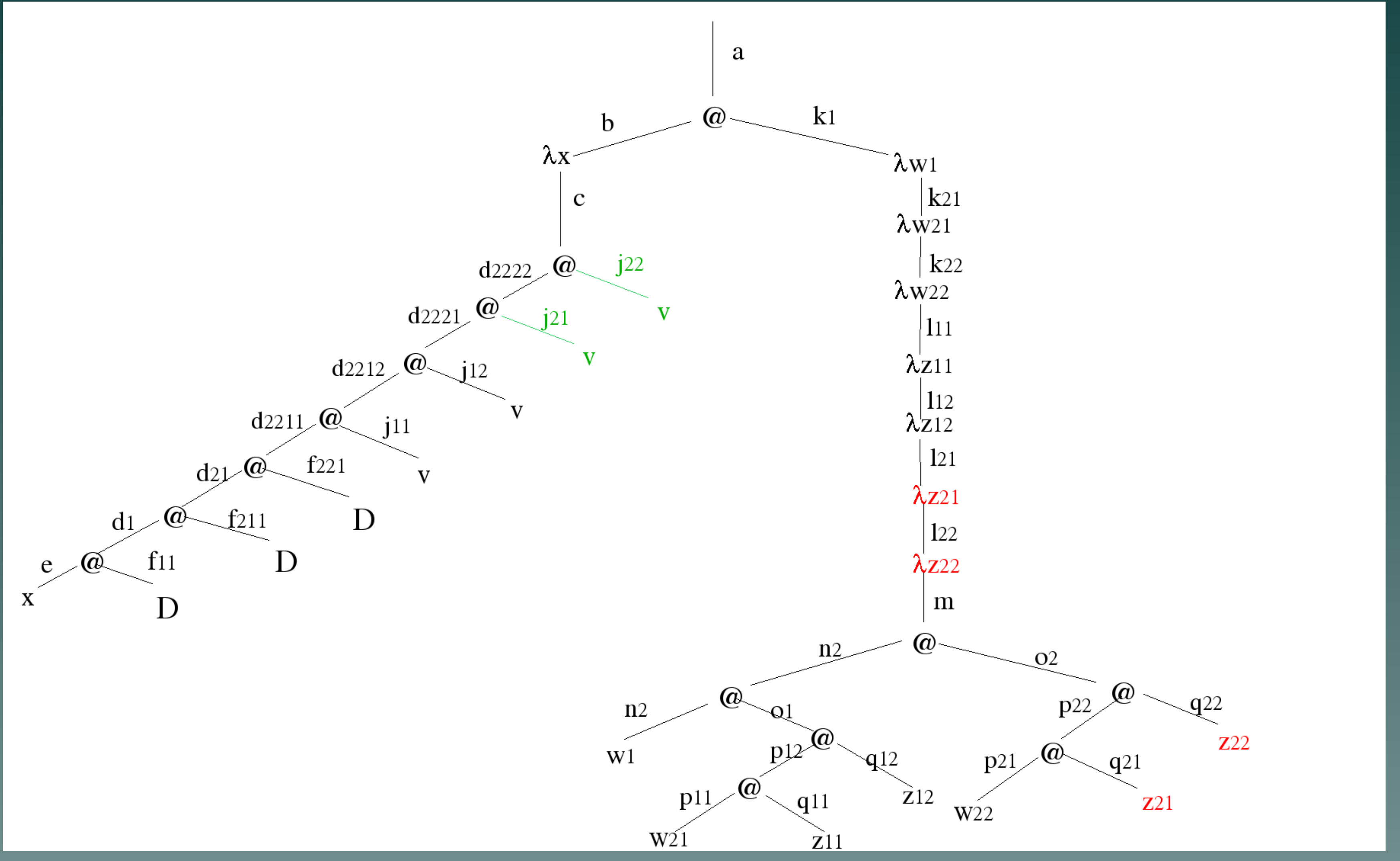

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Example

$$\mathcal{T}(\lambda x.x(\lambda y.yy)v)(\lambda fz.f(fz)) =$$

$$(\lambda x.xDDDvvvv)(\lambda w_1 w_2 w_3 z_1 z_2 z_3 z_4.w_1(w_2 z_1 z_2)(w_3 z_3 z_4))$$

Example

$$\mathcal{T}(\lambda x.x(\lambda y.yy)v)(\lambda fz.f(fz)) =$$

$$(\lambda x.xDDDvvvv)(\lambda w_1 w_2 w_3 z_1 z_2 z_3 z_4.w_1(w_2 z_1 z_2)(w_3 z_3 z_4))$$

$$(\lambda x.xDDDvvvv)(\lambda w_1 w_2 w_3 z_1 z_2 z_3 z_4.w_1(w_2 z_1 z_2)(w_3 z_3 z_4)) \rightarrow_{\beta^*} (vv)(vv)$$

Example

Let $\Delta = \lambda x.xx$, $D = \lambda x_1 x_2.x_1 x_2$, and $\Omega = \Delta\Delta$. We have:

$$\begin{aligned}
\mathcal{T}(\Omega) &= \mathcal{T}(D\Delta\Delta) = \\
&= \mathcal{T}(\lambda x_1 x_2.x_1 x_2 x_2) D\Delta) = \\
&= \mathcal{T}(\lambda x_1 x_2 x_3.x_1 x_2 x_3) D\Delta\Delta) = \\
&= \mathcal{T}(\lambda x_1 x_2 x_3.x_1 x_2 x_3 x_3) DD\Delta) = \\
&= \mathcal{T}(\lambda x_1 x_2 x_3 x_4.x_1 x_2 x_3 x_4) DD\Delta\Delta) = \cdots
\end{aligned}$$

Since the set of legal paths of Ω is not finite, $\mathcal{T}(\Omega)$ never terminates.

Conclusions

- Definition of the Weak Linear Lambda Calculus
- Transformation of general terms into weak linear terms
- Use of legal paths in term transformation