Locally Boolean Domains

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Abstract

We describe a notion of partial order with an involutive negation, and show that it gives rise to a cartesian closed category of bistable functions. We refine our notion of order to a form of dI-domain, and prove that all such "locally boolean domains" can be generated up to isomorphism by four operations; lifting, and taking products, co-products, and limits/co-limits of ω-chains. We define categories of locally boolean domains and affine functions, and pointed domains and strict affine functions, and prove that they are dual. We use this duality to describe a model of intuitionistic (affine) linear type theory, as an adjunction between an SMCC of locally boolean domains and affine functions and a CCC of locally boolean domains and bistable functions.

We also consider a simple notion of two-player game, and show that each such game gives rise to a locally boolean domain, and vice-versa. We show that the categories of games and strategies, and locally boolean domains and affine functions are equivalent. We further show that this equivalence preserves the structure of the categories as models of linear logic, and hence that the the cartesian closed category of locally boolean domains and bistable functions is equivalent to a CCC of sequential algorithms on sequential data structures with a single error.

Locally boolean domains may be seen as a particular kind of "bistable biorders" with close connections with game semantics and sequential algorithms. Bistability [9] is a variant of the stable ordering [2] which allows a CCC of functions which preserve both bounded meets and joins to be defined. Bistable biorders combine the bistable order with an extensional order (in the style of Berry's bidomains [2]), with the significant property that monotone and bistable functions are sequential. The key universality result for bistable biorders is that they provide a universal and fully abstract model of the λ-calculus with both bottom and top elements (a semantic description of the "minimal model" first characterized (syntactically) by Padovani [16]). This leads to full abstraction results for languages including SPCF, providing a link to previous, more intensional models of this language, such as "observably sequential functions" [4], and sequential algorithms [3, 5].

It is fairly straightforward to show that sequential algorithms over sequential data structures with a single error act as bistable functions on their arguments, and Curien [7] has shown that the category of sequential data structures and sequential algorithms embeds *fully* in the category of bistable bidomains and bistable functions. However, this leaves open the question of how the correspondence works in the opposite direction; what is the image of the embedding, and given an object in that image, can we construct the corresponding sequential data structures and sequential algorithms? The underlying linear structure of the sequential algorithms model also suggests a further question; it is equivalent to the co-Kleisli category of a certain comonad [11, 6] on a symmetric monoidal category of games and strategies. Can we provide a parallel "linear decomposition" of a CCC of bistable domains, and describe a purely extensional model of linear logic equivalent to the games model? Once this has been achieved, we may be in a position to reconstruct a substantial portion of game semantics, by finding preservation properties for functions which correspond to conditions on games and strategies, leading to further full completeness and full abstraction results.

Our constructions are based on a notion of "locally boolean" order — a partial order (the extensional order) with an involutive negation, which is not necessarily a boolean algebra, principally because $x \vee \neg x = \top$ fails in general. Using negation and the extensional order we are able to derive a simple definition of the stable order (and its dual, the co-stable order) which is equivalent to the standard one, and show that there is a CCC of locally boolean orders and bistable functions which embeds fully in the CCC of bistable orders and bistable biorders. Moreover, by adding the requirement to be a dI-domain with respect to the stable order, we arrive at a notion of *locally boolean domain*. Our fundamental representation result for these domains is that they can all be generated (up to isomorphism) by taking products and co-products, lifting, and limits of ω-chains. In particular, each pointed domain is isomorphic to one of the form $\Sigma_{i \in I}\Pi_{j \in J_i} A_{ij}$ which can naturally be read as a game in which Opponent chooses an index i in I and Player chooses an index j in J_i, followed by a dialogue in A_{ij}. This leads to an equivalence with a symmetric monoidal category of games which is essentially the Abramsky-Jagadeesan games model of the negative fragment of linear logic [1], and which corresponds to the affine sequential algorithms on sequential data structures [6]. Thus by describing a monoidal co-monad on the linear category of locally boolean domains, and showing that it is equivalent to the Lamarche ! on the category of games, we are able to show equivalence of the CCCs of locally boolean domains and bistable functions, and sequential algorithms on sequential data structures. Hyland and Schalk [8] have also given a representation of an equivalent category based on a more abstract notion of games, without moves, in which composition of strategies is simply composition of the underlying relations — this might be seen as intermediate between the games/sequential algorithms models and locally boolean domains, in that some intensional information is supplied explicitly with morphisms in the abstract games model, but less than in the token games model as there are in general several ways to reach a given position.

This work has benefitted greatly from discussions with Thomas Streicher and Pierre-Louis Curien.

1 Locally Boolean Orders[1]

Definition 1.1 *An involutive* negation operator *on a partial order* $(P, \leq)$ *is an endofunction* $\neg : P \rightarrow P$ *which is involutive —* $\neg\neg x = x$ *— and antitone — if* $x \leq y$*, then* $\neg y \leq \neg x$*.*
A locally boolean order $A = (|A|, \leq, \neg)$ *is a partial order with a negation operator such that:*

- *for each* x*,* $\{x, \neg x\}$ *has a least upper bound, which we shall write* $x^{\top}$ *(and thus* $\{x, \neg x\}$ *has a greatest lower bound* $x_{\perp} = \neg(x^{\top})$*).*
- *We shall write* $x \uparrow y$ *if* $x \leq y^{\top}$ *and* $y \leq x^{\top}$ *and require that if* $x \uparrow y$*, then* $\{x, y\}$ *has a greatest lower bound* $x \wedge y$ *and a least upper bound* $x \vee y$*, and if* $x \uparrow y$*,* $x \uparrow z$ *and* $y \uparrow z$ *then* $x \wedge (y \vee z) = (x \wedge y) \vee x \wedge z)$ *and* $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$*.*

The most basic examples of lbo are the empty order $\varnothing$, the one-point order $*$, and the two-point order $\Sigma = (\{\top, \perp\}, \leq, \neg)$ with $\top \leq \perp$ and $\neg\perp = \top, \neg\top = \perp$.

[1] In essence, locally boolean orders correspond to Streicher's "bistable bidomains with involution" [17, 14].

For any subset $X \subseteq |A|$, we shall write $\uparrow X$ if $x \uparrow y$ for all $x, y \in X$. If X is finite and non-empty then $\uparrow X$ implies that X has a glb $\bigwedge X$ and lub $\bigvee X$.

Definition 1.2 *A locally boolean order A is* complete *if for any non-empty $X \subseteq A$ such that $\uparrow X$, the lub $\bigvee X$ exists, and whenever $\uparrow (X \cup \{y\})$, then $(\bigvee X) \wedge y = \bigvee \{x \wedge y \mid x \in X\}$.*

A is pointed *if the empty set has a glb and lub — i.e. A has a least element $\bot$ and a greatest element $\top$ such that $\top = \neg\bot$.*

There is a dual to $\uparrow$ — $x \downarrow y$ if $\neg x \uparrow \neg y$, or $x_\bot \leq y$ and $y_\bot \leq x$. By involutivity of negation, if $x \downarrow y$ then x and y have least upper bound $\neg(\neg x \wedge \neg y)$ and a greatest lower bound$\neg(\neg x \vee \neg y)$. We shall write $x \updownarrow y$ if $x \uparrow y$ and $x \downarrow y$.

Lemma 1.3 *If $x \uparrow y$, then $x \uparrow x \vee y$ and $x \uparrow x \wedge y$.*

Proof: If $y \leq x^\top$ and $x \leq y^\top$ then $x \vee y \leq x^\top$, and $x \leq (x \vee y)^\top = x \vee y \vee \neg(x \vee y)$. And $x \uparrow x \wedge y$, since $\downarrow \{(\neg x \vee \neg y), x, y\}$, so $(x \wedge y)^\top = (x \wedge y) \vee (\neg x \vee \neg y) = (x^\top \vee \neg y) \wedge (y^\top \vee \neg x) \geq x$. $\Box$

Lemma 1.4 *$x \updownarrow y$ if and only if $x^\top = y^\top$.*

Proof: If $x \uparrow y$ and $x \downarrow y$, then $x \leq y^\top$, and $\neg(x) \leq (\neg(y))^\top = y^\top$, so $x^\top \leq y^\top$, and vice-versa. $\Box$

We can define three refinements to the extensional order; the stable, co-stable and bistable order, as follows.

Definition 1.5 *If A is a locally boolean order, the* stable *order, $\leq_s$ on $|A|$ is defined: $x \leq_s y$ if $x \uparrow y$ and $x \leq y$. The co-stable order on $|A|$ is defined $x \leq_c y$ if $x \downarrow y$ and $x \leq y$. The bistable order $\leq_b$ on $|A|$ is defined $x \leq_b y$ if $x \leq y$ and $x \updownarrow y$ (i.e. if $x \leq_s y$ and $x \leq_c y$).*

Lemma 1.6 *In a locally boolean order, $x \leq_s y$ if and only if $a \leq b \leq a^\top$ if and only if $a \leq b$ and $a_\bot \leq b_\bot$.*

Since we have $x \leq_c y$ if and only if $\neg y \leq_s \neg x$, we have $x \leq_c y$ if and only if $y_\bot \leq x \leq y$ if and only if $x \leq y$ and $x^\top \leq y^\top$.

Lemma 1.7 *If $x \uparrow y$, then $x \vee y$ and $x \wedge y$ are the least upper bound and greatest lower bound of $\{x, y\}$ with respect to the stable order.*

Proof: Suppose $x \uparrow y$. Then $x \wedge y \leq_s x, y \leq_s x \vee y$ by Lemma 1.3. Suppose $x, y \leq_s z$. Then $x \vee y \leq z$, and $z \leq (x^\top \vee y) \wedge (y^\top \vee x) = (x \vee y \vee \neg x) \wedge (x \vee y \vee \neg y) = (x \vee y) \vee (\neg x \wedge \neg y) = (x \vee y)^\top$. (Since $\downarrow \{(x \vee y), \neg x, \neg y\}$.) And if $z \leq_s x, y$, $z \leq x \wedge y \leq z^\top$. $\square$

Also, if x, y are bounded above in the stable order, we have $x \uparrow y$.

Lemma 1.8 *If A is a locally boolean order, then $(|A|, \leq, \leq_s)$ is a distributive biorder in the sense of Berry [2], and $(|A|, \leq, \leq_b)$ is a bistable biorder [9].*

We can also prove the following "interpolation lemma" for locally boolean orders.

Lemma 1.9 *Suppose $x \leq y$, then there exists z such that $x \leq_s z \leq_c y$.*

Proof: Take $z = x \vee y_\perp$. This is well defined since $x \uparrow y_\perp$, as $x \leq y$ implies $x \leq (y_\perp)^\top = y^\top$, and $y_\perp \leq x^\top$ (as $\neg y \leq \neg x$). Then $x \leq x \vee y_\perp \leq x^\top$ and $y_\perp \leq x \vee y_\perp \leq y$ as required. $\square$

Definition 1.10 *The product and co-product of an indexed family of lbos $\{A_i \mid i \in I\}$ are lbos defined pointwise — i.e. $\Pi_{i \in I} A_i = (\Pi_{i \in I}|A_i|, \leq, \neg)$ (where $\langle x_i \mid i \in I\rangle \leq \langle y_i \mid i \in I\rangle$ if $x_i \leq y_i$ for all i, and $\neg\langle x_i \mid i \in I\rangle = \langle \neg x_i | \; i \in I\rangle$).*
$\coprod_{i \in I} A_i = (\coprod_{i \in I} |A_i|, \leq, \neg)$ (where $\mathtt{in}_i(x) \leq \mathtt{in}_j(y)$ if $i = j$ and $x \leq y$, and $\neg(\mathtt{in}_i(x)) = \mathtt{in}_i(\neg(x))$).

Definition 1.11 *If A is a locally boolean order, then its* lift *$A_\uparrow$ is the pointed lbo with $|A_\uparrow| = \{\mathtt{in}(x) \mid x \in |A|\} \cup \{\perp, \top\}$, $\perp \leq x \leq \top$ for all x, and $\neg \top = \perp, \neg \perp = \top$.*
The separated sum of an indexed family of lbos is defined $\Sigma_{i \in I} A_i = (\coprod_{i \in I} A_i)_\uparrow$.

Definition 1.12 *The coalesced sum of an indexed family of pointed lbos is defined $|\bigoplus_{i \in I} A_i| = (\coprod_{i \in I}(|A_i| - \{\top, \perp\}) \cup \{\top, \perp\}$, $x \leq y$ if $x = \perp$ or $y = \top$ or $x = \mathtt{in}_i(x'), y = \mathtt{in}_i(y')$ and $x' \leq y'$, and $\neg(\top) = \perp$, $\neg(\perp) = \top$, and $\neg(\mathtt{in}_i(x)) = \mathtt{in}_i(\neg(x))$.*

Given lbos A, B and a function $f|A| \to |B|$ we say f is *monotone* if for all $x, y \in |A|$, $x \leq y$ implies $f(x) \leq f(y)$.

Definition 1.13 *For any function $f : |A| \to |B|$, we may define a function $\neg f : |A| \to |B|$ by $(\neg f)(x) = \neg(f(\neg x))$.*

If A, B are complete, then $f : A \to B$ is *continuous* if it preserves lubs of all non-empty and (upwards) $\leq$-*directed* sets X such that $\uparrow X$ (equivalently, lubs of all upwards $\leq_s$ directed sets), and glbs of all non-empty and (downwards) $\leq$-*directed* sets such that $\downarrow X$ (equivalently, glbs of all downwards $\leq_c$ directed sets).

We shall say that f is *boundedness-preserving* if for all x, y $x \uparrow y$ implies $f(x) \uparrow f(y)$ and $x \downarrow y$ implies $f(x) \downarrow f(y)$.

Definition 1.14 *If f, g are both boundedness-preserving then we may define functions $f \vee g$ and $f \wedge g$ pointwise: $(f \vee g)(x) = f(x) \vee g(x)$ and $(f \wedge g)(x) = f(x) \wedge g(x)$.*

Lemma 1.15 *Let $f : |A| \to |B|$ be a function on locally boolean orders. Then the following are equivalent.*

- *(i) f is monotone (continuous) and boundedness-preserving,*
- *(ii) f is monotone (continuous) and for all x, $(f(x^\top))^\top = (f(x))^\top$,*
- *(iii) f is monotone (continuous) with respect to the stable and co-stable orders.*

Proof: For $(i) \implies (ii)$, note that $x \updownarrow x^\top$, and so $f(x) \updownarrow x^\top$, and so $f(x) \updownarrow f(x^\top)$.

For $(ii) \implies (iii)$ suppose $x \leq_s y$, then $f(x) \leq f(y) \leq f(x^\top) \leq f(x^\top)^\top = (f(x))^\top$, and so f is monotone w.r.t. the stable order, whilst $(f(x_\perp))^\top = (f((x_\perp)^\top))^\top = (f(x^\top))^\top = (f(x))^\top$ and hence $(f(x_\perp))_\perp = (f(x))_\perp$ and so the same applies to the co-stable order.

For $(iii) \implies (i)$, note that if f is monotone w.r.t. the stable and co-stable order, then it must preserve stable and co-stable boundedness. It is monotone w.r.t. the extensional order, by the interpolation lemma. $\Box$

Definition 1.16 *We define further constraints on continuous and boundedness-preserving functions, as follows:*

- *f is* bistable *if it preserves bistably bounded binary lubs and glbs — i.e. if $x \updownarrow y$ then $f(x \vee y) = f(x) \vee f(y)$ and $f(x \wedge y) = f(x) \wedge f(y)$.*

- f *is* affine *if it preserves stably and co-stably bounded lubs and glbs — i.e. if* $x \uparrow y$ *or* $x \downarrow y$*, then* $f(x \vee y) = f(x) \vee f(y)$ *and* $f(x \wedge y) = f(x) \wedge f(y)$*.*

- *If* A, B *are pointed, then* f *is* strict *if it preserves empty lubs and sups — i.e.* $f(\bot) = \bot$ *and* $f(\top) = \top$*.*

- f *is* linear *if it is continuous, affine and strict (i.e. preserves glbs and lubs for all bounded sets).*

- f *is* total *if it is affine and preserves negation — i.e.* $f(\neg x) = \neg(f(x))$*.*

Note that if f is (monotone, bistable, affine, continuous, strict, total) then $\neg f$ is (monotone, bistable, affine, continuous, strict, total).

Lemma 1.17 *If* f *is affine, then* f *is continuous if and only if for all sets* X *such that* $\uparrow X$*,* $f(\bigvee X) = \bigvee f(X)$*, and for all sets such that* $\downarrow X$*,* $f(\bigwedge X) = \bigwedge f(X)$*.*

Using these notions we may define least upper bounds of ω-chains of complete lbos.

Definition 1.18 *We shall write* $A \trianglelefteq B$ *if there is a retraction from* A *to* B *— an affine function* $\phi^A_B : A \to B$ *and a total function* $\psi^B_A : B \to A$ *such that* $\psi^B_A \cdot \phi^A_B$ *is the identity on* A*, and* $\phi^A_B \cdot \psi^B_A \leq_s \mathtt{id}_A$*.*

An ω-chain of complete lbos is a family $\{A_i \mid i \in \omega\}$ such that $A_i \trianglelefteq A_{i+1}$ for all i. An upper bound for $\{A_i \mid i \in \omega\}$ is a complete lbo B, with retractions $A_i \trianglelefteq B$ for each i such that $\psi^{A_{i+1}}_{A_i} \cdot \psi^B_{A_{i+1}} = \psi^B_{A_i}$. We shall write $\{A_i \mid i \in \omega\} \trianglelefteq B$.

Definition 1.19 *Given an* ω*-chain of complete lbos we define a complete lbo,* $\bigsqcup_{i \in \omega} A_i$ *as follows:*
$|\bigsqcup_{i \in \omega} A_i| = \{x \in \Pi_{i \in \omega}|A_i| \mid \forall j \in \omega. \psi_{j+1}(\pi_{j+1}(x)) = \pi_j(x)\}$
$x \leq y$ *if for* $j \in \omega$*,* $\pi_j(x) \leq \pi_j(y)$
$\neg\langle x_i \mid i \in \omega\rangle = \langle \neg x_i \mid i \in \omega\rangle$

Proposition 1.20 $\bigsqcup_{i \in \omega} A_i$ *is a least upper bound for* $\{A_i \ i \in \omega\}$ *— i.e* $\{A_i \mid i \in \omega\} \trianglelefteq \bigsqcup_{i \in \omega} A_i$ *and if* $\{A_i \mid i \in \omega\} \trianglelefteq B$ *then* $\bigsqcup_{i \in \omega} A_i \trianglelefteq B$*.*

1.1 Locally Boolean Orders and Bistable Biorders

Let $\mathcal{LB}$ be the category of locally boolean orders and bistable functions, and let $\mathcal{CLB}$ be the category of locally boolean orders and continuous and bistable functions.

Proposition 1.21 *The operation which sends A to $(|A|, \leq, \leq_b)$ is a full embedding of $\mathcal{LB}$ in the category of bistable biorders and bistable functions.*

Proof: This is direct from the definition of the category of bistable biorders in [9]. $\square$

Moreover, $\mathcal{LB}$ has the same cartesian closed structure as the category of bistable biorders.

Lemma 1.22 *If f is bistable then $f^\top$ is bistable.*

Proof: For all x, $f(x) \uparrow (\neg f)(x)$, and so $f^\top = f \vee \neg f$ is well-defined. We need to show that it is bistable.
$(f^\top(x^\top)) = (fx) \vee f(\neg x) \vee \neg f(\neg x) \vee \neg(fx) = (fx)^\top \vee (f(\neg x))^\top = (f(x))^\top$, and hence $((f^\top(x^\top)))^\top = (f(x))^\top$ whilst $(f^\top(x))^\top = f(x) \vee \neg(f(\neg(x))) \vee (\neg(f(x)) \wedge f(\neg x)) = (f(x))^\top \vee \neg(f(\neg x))) \wedge ((f(\neg x))^\top \vee f(x) = (f(x))^\top$ as required. To show that $f^\top$ preserves bistably bounded glbs, observe that $f(a) \vee f^\top(a_\bot) = f(a) \vee f(a_\bot) \vee \neg(f(a^\top)) = f(a) \vee (\neg(f(\neg a)) \wedge \neg(f(a))) = f^\top(a) \vee (f(a))^\top = f^\top(a)$. Now suppose $x^\top = y^\top$, and so $(x \wedge y)_\bot = x_\bot = y_\bot$. Then $f^\top(x \wedge y) = f(x \wedge y) \vee f^\top((x \wedge y)_\bot) = f(x) \wedge f(y) \vee f^\top((x \wedge y)_\bot) = (f(x) \vee f^\top(x_\bot)) \wedge (f(y) \vee f(y_\bot)) = f^\top(x) \wedge f^\top(y)$. $\square$

Thus the stable order on bistable functions — $f \leq_s g$ if and only if $f \leq_s g \leq f^\top$ — is well-defined, and we can show that it corresponds to the usual definition.

Proposition 1.23 *$f \leq_s g$ if and only if $f(x) \leq_s g(x)$ for all x, and for all $x \leq_s y$, $f(x) = g(x) \wedge f(y)$ (if and only if for all $x \leq_b y$, $f(x) = g(x) \wedge f(y)$).*

Proof: Suppose $f \leq_s g$. Then $f(x) \leq g(x) \leq (f(x^\top)) \leq (f(x))^\top$, and if $x \leq_s y$, then $f(x) = f(x^\top) \wedge f^\top(x) \geq f(y) \wedge g(x) \geq f(x)$.

Suppose $f(x) \leq_s g(x)$ for all x, and for all $x \leq_b y$, $f(x) = g(x) \wedge f(y)$. Then $f(x) \leq g(x)$ for all x, and $f^\top(x) = f(x) \vee \neg f(\neg x) = (f(x^\top) \wedge g(x)) \vee \neg f(\neg x) = (f(x) \vee f(\neg x) \vee \neg f(\neg x)) \vee (g(x) \vee \neg f(\neg x) \geq (f(\neg x))^\top \wedge g(x) = (f(x))^\top \wedge g(x) = g(x)$. $\square$

Proposition 1.24 *$\mathcal{LB}$ and $\mathcal{CLB}$ are bi-cartesian closed.*

PROOF: $\mathcal{LB}$ has products and co-products as per Definition 1.10. We have already shown that $\neg\_$ and $(\_)^\top$ are well defined on functions, so to show that the function-space is a locally boolean order, it suffices to show that if $f \uparrow g$, then $f \vee g$ and $f \wedge g$ are bistable. $f \leq g^\top$, and so for all x, $f(x) \leq g(x) \vee \neg g(\neg x) \leq (g(x))^\top$, since $g(x) \leq (g(x))^\top$, and $(g(x))_\perp) \leq g(\neg x)$ so $\neg g(\neg x) \leq (g(x))^\top$. $\downarrow \{f(x) \vee g(x), \neg f(x), \neg g(x)\}$, and so $((f \vee g)(x))^\top = f(x) \vee g(x) \vee (\neg f(x) \wedge \neg g(x)) = ((f(x))^\top \vee g(x)) \wedge ((g(x))^\top \vee f(x)) = ((f(x))^\top \wedge (g(x))^\top$. Thus $((f \vee g)(x^\top))^\top = ((f(x^\top))^\top \wedge (g(x^\top))^\top = ((f(x))^\top \wedge (g(x))^\top = ((f \vee g)(x))^\top$.

Now suppose $x \updownarrow y$, then $f(x \wedge y) = f^\top(x \wedge y) \wedge f(x) = f(x) \wedge f^\top(y) \geq f(x) \wedge g(y)$, and similarly $f(x) \wedge f(y) \geq f(y) \wedge g(x)$. Hence $(f \vee g)(x) \wedge (f \vee g)(y) = (f(x) \wedge f(y)) \vee (g(x) \wedge g(y)) \vee (f(x) \wedge g(y)) \vee (f(y) \wedge g(x)) = f(x \wedge y) \vee g(x \wedge y)$. Now we can use the fact that $f, g \leq_s f \vee g$, and hence $f(x) = f(x \vee y) \wedge (f \vee g)(x)$, and $g(x) = g(x \vee y) \wedge (f \vee g)(x)$ Thus $(f \wedge g)(x) = (f \wedge g)(x \vee y) \wedge (f \vee g)(x)$, and similarly, $(f \wedge g)(y) = (f \wedge g)(x \vee y) \wedge (f \vee g)(y)$, and $(f \wedge g)(x) \vee (f \wedge g)(y) = (f \wedge g)(x \vee y) \wedge (f \vee g)(x \vee y) = (f \wedge g)(x \vee y)$ as required.

$\square$

So the full embedding of $\mathcal{LB}$ into the category of bistable biorders preserves CCC structure. $\mathcal{LB}$ therefore contains a universal and fully abstract model of the simply-typed λ-calculus with two constants, $\top$ and $\perp$ [9].

2 Locally Boolean Domains

We now place algebraicity conditions on locally boolean orders, to arrive at a notion of *locally boolean domain*. An element p of a locally boolean order is a *prime* if it is prime with respect to the stable order – i.e. for every a, b such that $a \uparrow b$, $p \leq_s a \vee b$ implies $p \leq_s a$ or $p \leq_s b$. An element c is co-prime if $\neg c$ is prime (i.e. c is prime w.r.t. the co-stable order).We shall write $P(A)$ for the set of primes in A, and $CP(A)$ for the compact primes, $P(x)$ for the set $\{p \in P(A) \mid p \leq_s x\}$, and $CP(x)$ for $P(x) \wedge CP(A)$.

Similarly, by *compact*, we shall mean compact with respect to the stable order.

Definition 2.1 *A* locally boolean domain *is a complete locally boolean order which is prime algebraic — i.e. $x = \bigvee CP(x)$ for all x — a dI-domain —*

i.e. every compact element dominates finitely many elements (in the stable order) — with the following two additional properties:

- *If p is prime and $p \leq_s a \vee b$, where $a \downarrow b$, then $p \leq_s a$ or $p \leq_s b$.*
- *The extensional order is determined by the primes in the following sense:*

$$x \leq y \iff \forall p \in P(x). \exists q \in P(y). p \leq q$$

Lemma 2.2 *In a locally boolean domain, p is prime if and only if for all a, b such that $a \uparrow b$ or $a \downarrow b$, $p \leq a \vee b$ implies $p \leq a$ or $p \leq b$.*

PROOF: Suppose p is prime and $p \leq a \vee b$, then there exists $q \in P(a \vee b)$ such that $p \leq q$. Hence $q \leq_s a$ or $q \leq_s b$ and so $p \leq a$ or $p \leq b$ as required. If p has the proposed property, and $p \leq_s a \vee b$ (where $a \uparrow b$) then $p \leq x \vee y \leq p^\top$ and hence $p \leq x \leq p^\top$ or $p \leq y \leq y^\top$ as required. □

Note that an element x of a lbd is compact (with respect to the stable order) if and only if $P(x)$ is finite. For compact x, we shall write $\#(x)$ for $|P(x)|$.

Lemma 2.3 *For any family of locally boolean domains, $\{A_i \mid i \in I\}$, the product $\Pi_{i \in I} A_i$ and co-product $\coprod_{i \in I} A_i$ are locally boolean domains, and if each A_i is pointed, then the coalesced sum $\bigoplus_{i \in I} A_i$ is a locally bolean domain. If $\{A_i \mid i \in \omega\}$ is a ω-chain of locally boolean domains, then $\bigsqcup_{i \in \omega} A_i$ is a locally boolean domain. For any locally boolean domain A, the lift $A_\uparrow$ is a locally boolean domain.*

PROOF: The sets of primes are as follows:
$P(A \times B) = \{\langle p, \bot \rangle \mid p \in A\} \cup \{\langle \bot, p \rangle \mid p \in B\}$.
$P(A + B) = P(A) + P(B)$.
$P(\bigsqcup_{i \in \omega} A_i) = \bigcup_{j \in \omega} \{\phi(p) \mid p \in P(A_j)\}$
$P(A_\uparrow) = P(A) + \{\bot, \top\}$. □

Moreover, every locally boolean domain can be expressed using products, coproducts, lifting and taking limits, as we shall now show.

Definition 2.4 *For a locally boolean domain A, let $\mathsf{m}(A)$ be the set of elements of A which are minimal w.r.t. $\leq_s$.*

Note that $\mathsf{m}(a)$ is empty only if A is empty, since otherwise there is an infinite descending chain of primes in A, contradicting the fact that it is a dI-domain.

Lemma 2.5 *If A is a locally boolean domain, then there is a family $\{A_x \mid x \in \mathsf{m}(A)\}$ of pointed locally boolean orders domains such that $A \cong \coprod_{x \in \mathsf{m}(A)} A_x$.*

PROOF: For every element $a \in A$ there exists $x_a \in \mathsf{m}(A)$ such that $x \leq_s a$. Moreover x_a is unique: suppose $x, y \leq_s a$. Then $x \wedge y \leq_s x$, and so $x = x \wedge y \leq_s y$ by minimality of x. So for each $x \in \mathsf{m}(A)$ we can define a locally boolean order A_x with $|A_x| = (\{x \in |A| \mid x \leq_s a\}, \leq_x = \leq_A \upharpoonright A_x$ and $\neg_x x = \neg(x)$ — so the key thing to check is that if $x \leq_s a$, then $x \leq_s \neg a$. Since there exists $y \in \mathsf{m}(A)$ such that $y \leq_s \neg x \leq_s x^\top$, we have $x, y \leq_s x^\top$, and hence $x = y$ as above.

Hence we can now show that $A \cong \coprod_{x \in \mathsf{m}(A)} A_x$ by the isomorphism $a \mapsto \mathtt{in}_x(a)$ if $x \leq_s a$. $\square$

Definition 2.6 *A locally boolean domain is said to be* lifted *if has a top element which is prime and distinct from $\bot$.*

Note that $A \times B$ is lifted if and only if $A = \mathbf{1}$ and B is lifted, or vice versa. The following lemma justifies the terminology.

Lemma 2.7 *A locally boolean domain A is lifted if and only if there is a locally boolean domain, $A^\downarrow$ such that $A = (A^\downarrow)_\uparrow$.*

PROOF: $A^\downarrow$ is defined in the obvious way, by removing the bottom and top elements of A — $A^\downarrow = (|A| - \{\bot, \top\}, \leq \upharpoonright |A^\downarrow|, \neg \upharpoonright |A^\downarrow|)$, so clearly $(A^\downarrow)_\uparrow = A$. It remains to check that $A^\downarrow$ is a locally boolean domain. If $x \in |A| - \{\bot, \top\}$, then $\neg x, x^\top \in |A| - \{\bot, \top\}$, since if $x^\top = x \vee \neg x = \top$, then $x = \top$ or $\neg x = \top$ by primality of $\top$, and so $x \in \{\top, \bot\}$. And if $x, y \in A^\downarrow$ and $x \uparrow y$ or $x \downarrow y$, then $x \vee y \in A^\downarrow$, since if $x \vee y = \bot$ then $x = y = \bot$ and if $x \vee y = \top$, then $x = \top$ or $y = \top$ by primality of $\top$. Similarly, $x \wedge y = \neg(\neg x \vee \neg y) \in A^\downarrow$. $\square$

Finally we can show that every pointed locally boolean domain can be (uniquely) decomposed as a product of lifted domains.

Lemma 2.8 *If X is a $\leq_s$-directed set of primes of a locally boolean domain, then $\bigvee X$ is prime.*

PROOF: Suppose $a \uparrow b$, and $\bigvee X \not\leq_s a$ and $\bigvee X \not\leq_s b$. Then there exists $p \leq_s X$ such that $p \not\leq_s a$, and hence there exists $p' \in X$ such that $p' \not\leq_s a$. Similarly there exists $q \in X$ such that $q \not\leq_s b$. By directedness, there exists $r \in X$ such that $p', q \leq_s r$ and so $r \not\leq_s a \vee b$ and $\bigvee X \not\leq_s x$ as required. $\square$

Lemma 2.9 *If A is a pointed locally boolean domain, then is an indexing set $\mathsf{p}(A)$ and a family of lifted locally boolean domains $\{A_p \mid p \in \mathsf{p}(A)\}$ such that $A \cong \Pi_{p\in\mathsf{p}(A)}A_p$.*

PROOF: Let $\mathsf{p}(A)$ be the set of $\leq_s$-maximal elements of $P(\top)$. By Zorn's Lemma this is non-empty, since every directed set in $P(\top)$ has an upper bound in $P(\top)$ by Lemma 2.8. So $\mathsf{p}(A)$ is stably bounded and $\bigvee \mathsf{p}(A) = \top$, and so $p \in \mathsf{p}(A)$ implies $p^\top = \top$. For each $p \in \mathsf{p}(A)$, we define a locally boolean domain A_p by letting $|A_p| = \{x \in |A| \mid x \leq p\}$, $\leq_p = \trianglelefteq A_p$ and $\neg_p(x) = p \wedge \neg(x)$, which is well-defined since $\neg x \downarrow p$: $p_\perp = \perp \leq \neg x$ and $(\neg x)_\perp = x_\perp \leq x \leq p$.

To verify that this is a locally boolean domain, the key fact to check is that $\neg_p$ is involutive: for any $x \leq p$, $\neg_p(\neg_p(x)) = p \wedge \neg(p \wedge \neg x) = p \wedge ((\neg p) \vee x) = p_\perp \vee (p \wedge x) = x$.

Note also that for distinct $p, q \in \mathsf{p}(A)$, $p \wedge q = \perp$, since $p \leq_s q \vee \neg q = \top$, so by primality and maximality of p, $p \leq_s \neg q$, hence $p \wedge q \leq_s \neg q \wedge q = q_\perp = \perp$. For any prime r in $P(A)$, $r \leq \top$ and so there exists $p \in \mathsf{p}(A)$ such that $r \leq p$, moreover, if $r \neq \perp$, then p is unique since if $r \leq p, p'$ and $p \neq p'$, then $r \leq p \wedge p' = \perp$. Also, if $p, q \in \mathsf{p}(A)$, $p \neq q$, $r \leq p$ and $r' \leq q$, then $r \uparrow r'$, since (assuming $r, r' \neq \perp$) $r' \leq p^\top = \top$ implies $r' \leq \neg(p) \leq \neg(r)$ (since $r' \not\leq p$) so $r' \leq r^\top$ and vice-versa.

Hence $A \cong \Pi_{p\in\mathsf{p}(A)}A_p$, by the order isomorphism which sends $x \in A$ to $\langle \bigvee\{q \leq p \mid q \leq_s x\} \mid p \in \mathsf{p}(A)\rangle$, and takes $\langle x_p \mid p \in \mathsf{p}(A)\rangle$ to $\bigvee\{r \in P(A) \mid \exists p \in \mathsf{p}(A).r \leq_s x_p\}$. A_p is lifted because it has at least two elements — if $p = \perp$, then $\perp$ is maximal in $P(\top)$ and so A has only one element — and its top element is prime because p is prime in A.

$\square$

By putting together Lemmas 2.5, 2.7 and 2.9 we can show that all locally boolean domains can be decomposed into sums and products.

Corollary 2.10 *If A is pointed, then $A \cong \Pi_{p\in\mathsf{p}(A)}\Sigma_{x\in\mathsf{m}(A_p^\downarrow)}A_{px}$.*

Definition 2.11 *Say that a locally boolean domain A is* bounded *if there is a natural number $\#(A)$ such that for all $p \in P(A)$, $\#(p) \leq \#(A)$.*

Lemma 2.12 *Given $\{A_i \mid i \in I\}$ such that each A_i is bounded, $\coprod_{i\in I} A_i$ and $\Pi_{i\in I}A_i$ are bounded, and $\#(\coprod_{i\in I} A_i) = \#(\Pi_{i\in I}A_i) = \max(\{\#(A_i) \mid i \in I\})$ and $\#(A_\uparrow) = \#(A) + 1$ (if A is non-empty).*

Proposition 2.13 *A locally boolean domain is bounded if and only if it is isomorphic to a domain generated from the product, co-product and lifting operations.*

By repeatedly applying Corollary 2.10, we can show that every pointed domain is isomorphic to a *prenex normal form* in which Π and Σ alternate, which is unique up to renaming of index sets. To obtain normal forms for unbounded domains we take limits of series of approximants.

Definition 2.14 *For each lbd A, we define a ω-chain $\{A^i \mid i \in \omega\}$ as follows: $A^0 = *$ and $A^{i+1} = \Pi_{p \in \mathsf{p}(A)} \Sigma_{x \in \mathsf{m}(A_p^{\downarrow})} A^i_{px}$.*

Lemma 2.15 $\{A_i \mid i \in \omega\} \trianglelefteq A$.

Lemma 2.16 *If $p \in P(A)$ and $\#(p) \leq n$ then there exists $q \in A_n$ such that $\phi_i(p) = q$.*

Proof: is by induction, using Corollary 2.10. $\square$

Theorem 2.17 (Decomposition Theorem) *If A is a pointed locally boolean domain, then $A \cong \bigsqcup_{i \in \omega} A^i$.*

Proof: $A_i \trianglelefteq A$ for all A, and hence there is a (structure preserving) injection $\phi : \bigsqcup_{i \in I} A_i \to A$, so it suffices to show that ϕ is surjective. To show this, it suffices to observe that by Lemma 2.16, and the fact that A is a dI-domain, for every compact prime p in A there exists $\phi^{-1}(p) \in \bigsqcup_{i \in \omega} A_i$ such that $\phi(\phi^{-1}(p)) = p$, since then for all $x \in A$, $x = \phi(\bigvee\{\phi^{-1}(p)) \mid p \in CP(A)\}$. $\square$

If we can prove that a property is preserved by taking products, co-products, liftings and limits, then the decomposition theorem entails that it applies to all locally boolean domains. For example, we can prove the reverse ("hard") version of the interpolation property, that all sets have a least upper bound and greatest upper bound with respect to the extensional order, and that for functions between locally boolean domains, monotone and bistable is equivalent to stable and co-stable.

Corollary 2.18 *Given $x \leq y$ there exists z (in general, not unique) such that $x \leq_c z \leq_s y$.*

Corollary 2.19 *Every locally boolean domain is a lattice with respect to $\leq$ (not, in general,* distributive*).*

Lemma 2.20 *Given x, y such that $x \uparrow y$, there exist x', y' such that $x \leq x'$, $y \leq y'$, $x' \updownarrow y'$ and $x' \wedge y' = x \wedge y$.*

Proof: This property is preserved by taking products, co-products, liftings, and limits. For example, suppose $x, y \in A_{\uparrow}$. Then if $x, y \notin \{\bot, \top\}$, we have $x', y' \in A$ fulfilling the above condition. Otherwise either $x = y = \top$ (so let $x' = y' = \top$) or $x \wedge y = \bot$ and either $x = \bot$ or $y = \bot$. In the first case, we let $x' = \bot$ and $y' = \top$, in the latter $x' = \top$ and $y' = \bot$. $\square$

Following [2], we may say that a function between locally boolean domains is *stable* if it is monotone (continuous) with respect to the stable order, and preserves greatest lower bounds of pairs of elements which are bounded above in the stable order. A function f is *co-stable* if $\neg f$ is stable.

Proposition 2.21 *A function between locally boolean domains is monotone and bistable if and only if it is stable and co-stable.*

Proof: If f is stable and co-stable, then it is monotone and boundedness-preserving by Lemma 1.15. Since it preserves stably bounded glbs and co-stably bounded lubs, it preserves bistably bounded glbs and lubs.

If f is monotone and bistable, then it is monotone w.r.t. the stable and costable orders by Lemma 1.15. Now suppose $x \uparrow y$. Then $f(x \wedge y) \leq f(x) \wedge f(y)$ and by Lemma 2.20 there exist $x' \geq x$ and $y' \geq y$ such that $x' \updownarrow y'$ and $x \wedge y = x' \wedge y'$ so $f(x \wedge y) = f(x' \wedge y') = f(x') \wedge f(y') \geq f(x) \wedge f(y)$. $\square$

3 A Model of Linear Logic

We shall now define a model of intuitionistic linear logic based on locally boolean domains, and affine and linear functions.

Definition 3.1 *We define the following categories of locally boolean domains:*

- *$\mathcal{LBA}$, the category of locally boolean domains and (continuous and) affine functions,*
- *$\mathcal{PLBA}_{\mathcal{S}}$, the category of pointed locally boolean domains and linear (i.e. strict and affine) functions,*

Proposition 3.2 *$\mathcal{LBA}$ and $\mathcal{PLBA}_\mathcal{S}$ have all small products and co-products, (co-products in $\mathcal{PLBA}_\mathcal{S}$ are coalesced sums).*

Lifting of locally boolean domains extends in an natural way to act as a functor on $\mathcal{LBA}$ and $\mathcal{PLBA}_\mathcal{S}$.

Lemma 3.3 *Lifting acts on $\mathcal{LBA}$ as a monad and on $\mathcal{PLBA}_\mathcal{S}$ as a co-monad.*

PROOF: Given $f : A \to B_\uparrow$, we define $f^* : A_\uparrow \to B_\uparrow$ by $f^*(\top) = \top$, $f^*(\bot) = \bot$ and $f^*(\mathtt{in}(x)) = f(x)$. Then $((\_)_\uparrow : \mathcal{LBA} \to \mathcal{LBA}, \mathtt{up} : I \to (\_)_\uparrow, (\_)^*)$ (where $\mathtt{up}(x) = \mathtt{in}(x)$) is a Kleisli triple.

Given strict $f : A_\uparrow \to B$ we define a strict function $f_* : A_\uparrow \to B_\uparrow$ by $f_*(\bot) = \bot$, $f_*(\top) = \top$, and $f_*(\mathtt{in}(x)) = \mathtt{in}(f(x))$. Then $((\_)_\uparrow : \mathcal{PLBA}_\mathcal{S} \to \mathcal{PLBA}_\mathcal{S}, \mathtt{down} : (\_)_\uparrow \to I, (\_)^*)$ (where $\mathtt{down}(\top) = \top$, $\mathtt{down}(\bot) = \bot$ and $\mathtt{down}(\mathtt{in}(x)) = x$) is a co-Kleisli triple. $\Box$

We now define two function-space operations on locally boolean domains.

Definition 3.4 *Given locally boolean domains A, B, we define $A \multimap B$ by letting $|A \multimap B|$ be the affine functions from A to B, with the standard extensional ordering and negation. If A, B are pointed, $A \multimap_s B$ is defined by letting $|A \multimap_s B|$ be the linear functions from A to B.*

$A \multimap B$ and $A \multimap_s B$ are partial orders with negation operators. To show that they are in fact locally boolean domains, we shall use the decomposition theorem to show that they can be reconstructed (up to isomorphism) using products, co-products, liftings and limits.

We first require a series of lemmas connecting $\multimap$ and $\multimap_s$ with the the sum, product, lifting and limit constructions.

Lemma 3.5 *The lifting co-monad on $\mathcal{PLBA}_\mathcal{S}$ resolves as the adjunction between the inclusion of $\mathcal{PLBA}_\mathcal{S}$ into $\mathcal{LBA}$ and lifting as a functor from $\mathcal{LBA}$ into $\mathcal{PLBA}_\mathcal{S}$.*

Corollary 3.6 *If B is pointed, $A_\uparrow \multimap_s B \cong A \multimap B$.*

Lemma 3.7 *If $f : A \to B_\uparrow$ is non-strict, non-$\bot$ and non-$\top$, then $f = \widehat{f}; \mathtt{up}$ for some $\widehat{f} : A \to B$.*

PROOF: We have $f(\top) \neq \top$ since $f(\top) \updownarrow f(\bot)$ but f is not strict or constantly $\top$, and similarly $f(\bot) \neq \bot$. Thus for all x, there is some $y \in B$ such that $f(x) = \mathtt{in}(y)$ and we can define $\widehat{f}(x) = y$. $\square$

Lemma 3.8 *If A is pointed, $A \multimap B_\uparrow \cong ((A \multimap_s B_\uparrow) + (A \multimap B))_\uparrow$.*

PROOF: If f is strict and g is non-strict, non-$\top$ and non-$\bot$, then $g \not\leq f$, and $f \not\leq g$. Thus $A \multimap B_\uparrow \cong ((A \multimap_s B_\uparrow) + (A \multimap B))_\uparrow$ by the isomorphism which sends $\bot$ to $\bot$, $\top$ to $\top$, each strict function f to $\mathtt{in_l}(f)$, and each non-strict, non-$\bot$ and non-$\top$ function to $\mathtt{in_r}(\hat{f})$. $\square$

Hence (for pointed A) we have $A \multimap \Sigma = A \multimap \varnothing_\uparrow \cong (A \multimap_s \Sigma)_\uparrow$.

We recall a key property of bistable functions from [9] (which must also apply to affine functions) — that they are *sequential* in the following sense.

Definition 3.9 *Let $\{A_i \mid i \in I\}$ and B be pointed locally boolean orders. A map $f : \Pi_{i \in I} A_i \to B$ is i-strict if $f(x) = \bot$ whenever $\pi_i(x) = \bot$ and $f(x) = \top$ whenever $\pi_i(x) = \bot$.*

Lemma 3.10 *Let $\{A_i \mid i \in I\}$ be a family of pointed locally boolean orders. If $f : \Pi_{i \in I} A_i \to C_\uparrow$ is a strict and* bistable *function, then f is i-strict for some unique $i \in I$.*

PROOF: Given $j \in I$ and $a \in A_j$, let $\bot[a]_j = \langle x_i \mid i \in I \rangle$, where $x_i = a$ if $i = j$ and $x_i = \bot$, otherwise, and let $\top[a]_j = \langle x_i \mid i \in I \rangle$, where $x_i = a$ if $i = j$ and $x_i = \top$, otherwise.

Then if $\pi_i(x) = \bot$, $x \leq \top[\bot]_i$, and if $\pi_i(x) = \top$, $\bot[\top]_i \leq x$, and so f is i-strict if $f(\top[\bot]_i) = \bot$, and $f(\bot[\top]_i) = \top$. Since $\top[\bot]_j \updownarrow \top[\bot]_k$ for all j, k, and $\bigwedge_{i \in I} \top[\bot]_i = \bot$, we have $f(\bigwedge_{i \in I} \top[\bot]_i) = f(\bot) = \bot$, and so $f(\top[\bot]_i) = \bot$ for some i, and similarly $f(\bigvee_{i \in I}(\bot[\top]_i) = \top$, and so $f(\bot[\top]_j) = \top$ for some j. Moreover, if $i \neq j$, then $\bot[\top]_j \leq \top[\bot]_i$, and so $i = j$ as required. $\square$

Lemma 3.11 *If $f : \Pi_{i \in I} A_i \to C_\uparrow$ is* affine*, and i-strict then $f = \pi_i; g$ for some $g : A_i \to C$.*

PROOF: Suppose $f : \Pi_{i \in I} A_i \to C_\uparrow$ is i-strict. Then for all $a \in A_i$, $\bot[a]_i \uparrow \top[\bot]_i$, and so $f(\top[a]_i) = f(\bot[a]_i \vee \top[\bot]_i) = f(\bot[a]_i) \vee \bot = f(\bot[a]_i)$. For all $x \in \Pi_{i \in I} A_i$, we have $\bot[\pi_i(x)]_i \leq x \leq \top[\pi_i(x)]_i$ and hence $f(x) = f(\bot[\pi_i(x)]_i)$. Thus $f = \pi_l; g$, where $g = \langle \mathtt{id}_A, \bot \rangle; f$. $\square$

Corollary 3.12 *For any family of pointed lbds, $\{A_i \mid i \in I\}$, $\Pi_{i\in I} A_i \multimap_s B_\uparrow \cong \coprod_{i\in i}(A_i \multimap_s B_\uparrow)$.*

Lemma 3.13 *If A is pointed, then $A \multimap \coprod_{i\in I} B_i \cong \coprod_{i\in I}(A \multimap B_i)$.*

PROOF: Given $f : A \to \coprod_{i\in I} B_i$, $f(\top) = \mathtt{in}_i(y)$ for some $i \in I$ and $y \in B_i$, and for all $x \in A$, $f(x) \leq \mathtt{in}_i(y)$, and so $f(x) = \mathtt{in}_i(z_x)$ for some $z_x \in B_i$. So we can define $\widehat{f} : A \to B_i$ such that $f = \widehat{f}; \mathtt{in}_i$ by letting $\widehat{f}(x) = z_x$. So $A \multimap \coprod_{i\in I} B_i \cong \coprod_{i\in I}(A \multimap B_i)$ by the isomorphism which sends f to $\mathtt{in}_i(\widehat{f})$. □

Lemma 3.14 *If $\{A_i \mid i \in \omega\}$ is a ω-chain of complete locally bolean orders, and B is a complete lbo, then $\bigsqcup_{i\in\omega} A_i \multimap B \cong \bigsqcup_{i\in\omega}(A_i \multimap B)$ and $B \multimap \bigsqcup_{i\in\omega} A_i \cong \bigsqcup_{i\in\omega}(B \multimap A_i)$.*

Proposition 3.15 *If A, B are locally boolean domains, then $A \multimap B$ is a locally boolean domain, and if A, B are pointed then $A \multimap_s B$ is a locally boolean domain.*

PROOF: We shall first prove that if A, B are bounded, then $A \multimap B$ and $A \multimap_s B$ are bounded locally boolean domains, by induction on $\#(A)+\#(B)$, for which the base case is trivial.

For the induction step, we first suppose A and B are pointed, and show that $A \multimap_s B$ and $A \multimap B$ are locally boolean domains. By the decomposition theorem, $A \cong \Pi_{i\in I}(A_i)_\uparrow$ where $\#(A_i) < \#((A_i)_\uparrow) \leq \#(A)$ for each i, and $B \cong \Pi_{j\in J}(B_j)_\uparrow$, where $\#(B_j) < \#((B_j)_\uparrow) \leq \#(B)$ for each j. Thus $A \multimap_s B \cong \Pi_{i\in I}(A \multimap_s (B_j)_\uparrow)$, and by Lemma 3.12, $A \multimap_s (B_j)_\uparrow = (\Pi_{i\in I}(A_i)_\uparrow) \multimap_s (B_j)_\uparrow \cong \coprod_{i\in I}((A_i)_\uparrow \multimap_s (B_j)_\uparrow)$. By Corollary 3.6, for each i, j we have $(A_i)_\uparrow \multimap_s (B_j)_\uparrow \cong A_i \multimap (B_j)_\uparrow$. By induction hypothesis, each $A_i \multimap (B_j)_\uparrow$ is a locally boolean domain, hence so is $A \multimap_s B$.

Now we show $A \multimap B$ is a locally boolean domain: $A \multimap B \cong \Pi_{j\in J}(A \multimap (B_j)_\uparrow)$, and by Lemma 3.8, $A \multimap (B_j)_\uparrow \cong ((A \multimap_s (B_j)_\uparrow) + (A \multimap B_j))_\uparrow$. We have already shown that each $A \multimap_s (B_j)_\uparrow$ is a locally boolean domain, and the inductive hypothesis implies that each $A \multimap B_j$ is a locally boolean domain. Hence so is $A \multimap B$.

To conclude the induction, we consider the case where A, B need not be pointed. Then $A \cong \coprod_{i\in I} A_i$ where each A_i is pointed and $\#(A_i) \leq \#(A)$, and $B \cong \coprod_{j\in J} B_j$, where each B_j is pointed and $\#(B_j) \leq \#(B)$. Then

$A \multimap B \cong \Pi_{i\in I}(A_i \multimap B) \cong \Pi_{i\in I}(\coprod_{j\in J}(A_i \multimap B_j))$, by Lemma 3.13. For each i, j, $A_i \multimap B_j$ is a locally boolean domain by the above argument for pointed lbds, and hence so is $A \multimap B$.

Finally, we apply Lemma 3.14 to show that $A \multimap B$ and $A \multimap_s B$ are locally boolean domains for any A, B. We have $A \cong \bigsqcup_{i\in\omega} A^i$ and $B \cong \bigsqcup_{j\in\omega} B^j$, and thus $A \multimap B \cong \bigsqcup_{i\in\omega}(\bigsqcup_{j\in\omega}(A^i \multimap B^j))$, and since $A^i \multimap B^j$ is a locally boolean domain for all i, j, so is $A \multimap B$ (and similarly $A \multimap_s B$). $\square$

We have the following lemma, proved by induction over the proof of Proposition 3.15.

Lemma 3.16 *The least upper bound and greatest lower bound of a set of functions F such that $\uparrow F$ or $\downarrow F$ is given pointwise — i.e. $(\bigvee F)(x) = \bigvee\{f(x) \mid f \in F\}$ and $(\bigwedge F)(x) = \bigwedge\{f(x) \mid f \in F\}$.*

We can also show that part of the definition of affineness for functions is redundant.

Lemma 3.17 *If $f : A \to B$ is monotone, boundedness-preserving, and preserves stably bounded lubs and glbs, then f is affine.*

Proof: First note that f is bistable, since f preserves all bistably bounded lubs and glbs, which are also stably bounded lubs and glbs. Hence by Proposition 2.20, f preserves co-stably bounded lubs. To show that f preserves co-stably bounded lubs, we observe that the proof of Proposition 3.15 (and the lemmas on which it is based) does not at any point use the fact that affine functions use co-stably bounded lubs. Hence we can re-prove the proposition using the weaker definition of $A \multimap B$ and $A \multimap_s B$ as spaces of monotone functions preserving boundedness and stably bounded lubs and glbs. $\square$

We may now show that $\_ \multimap \_$ and $\_ \multimap_s \_$ act as mixed variance functors on categories of locally boolean domains in various ways. In particular we shall make use of the following observation.

Proposition 3.18 *For any pointed locally boolean domain P, the functors $\_ \multimap P : \mathcal{LBA} \to \mathcal{PLBA_S}^{OP}$ and $\_ \multimap_s P : \mathcal{PLBA_S}^{OP} \to \mathcal{LBA}$ are adjoint — i.e. $\_ \multimap P \dashv \_ \multimap_s P$.*

Proof: We have natural transformations $\eta_A : A \to (A \multimap P) \multimap_s P$ and $\epsilon_A : A \to (A \multimap P) \multimap_s P$, both defined in the obvious way — i.e. $\eta(x) = f_x$, where $f_x(g) = g\,x$ — which are the unit and co-unit of the adjunction. $\square$

3.1 Locally Boolean Domains and Duality

In order to construct locally boolean models of (various frgaments of) linear logic, we shall show that the categories $\mathcal{LBA}$ and $\mathcal{PLBA}_{\mathcal{S}}$ are *dual*, via the adjunction $\_ \multimap \Sigma \dashv \_ \multimap_s \Sigma$ (which corresponds to the *Player/Opponent* duality in games). This result can, in fact, be proved inductively using the decomposition theorem and the facts we have already established about affine and linear morphisms, but we shall give a more direct proof.

For ease of manipulation we shall express elements of $A \multimap \Sigma$ as *conditionally prime filters* on A.

Definition 3.19 *Let A be a locally boolean domain, a conditionally prime filter over A is a subset $X \subseteq A$ such that:*

- *if $x \leq y$, and $x \in X$, then $y \in X$,*
- *if $x \uparrow y$ or $x \downarrow y$, and $x, y \in X$, then $x \wedge y \in X$,*
- *if $x \uparrow y$ or $x \downarrow y$, and $x \vee y \in X$, then $x \in X$ or $y \in X$.*

A conditionally prime filter X is strict if neither X nor its complement is empty.

Clearly, conditionally prime filters correspond bijectively with affine maps from A into Σ, and strict c.p. filters with linear maps into Σ. Thus we shall represent $A \multimap \Sigma$ as the locally boolean domain of c.p. filters of A, where $\leq$ is inclusion of filters, and negation is the operation $\neg X = \{x \in A \mid \neg x \notin X\}$. By Proposition 3.18, the functor which takes a locally boolean domain to its conditionally prime filters is right adjoint to the functor which takes a pointed lbd to its strict c.p. filters. The unit and co-unit of this adjunction are given by the maps $\eta(x) = \{y \in A \multimap \Sigma \mid x \in y\}$ and $\epsilon(x) = \{y \in A \multimap_s \Sigma \mid x \in y\}$.

Lemma 3.20 *Let A be a locally boolean domain. For each prime $p \in A$, there is a conditionally prime filter $\lfloor p \rfloor = \{x \in A \mid p \leq x\}$, which is prime in $A \multimap \Sigma$.*

Proof: For each prime p, $\lfloor p \rfloor$ is a prime, since if $X \uparrow Y$ or $X \downarrow Y$, then $\lfloor p \rfloor \leq X \vee Y$ iff $p \in X \vee Y$ iff $p \in X$ or $p \in Y$. $\square$

We also have co-primes $\lceil c \rceil = \{x \in A \mid x \not\leq c\}$ in $A \multimap \Sigma$ for each coprime c of A.

Lemma 3.21 *For all $p \in P(A)$, $\neg\lfloor p \rfloor = \lceil \neg p \rceil$, and for $p, q \in P(A)$, $p \leq q$ if and only if $\lfloor q \rfloor \leq \lfloor p \rfloor$.*

Lemma 3.22 *If A is a locally boolean domain, then $P(A \multimap \Sigma) = \varnothing \cup \{\lfloor p \rfloor \mid p \in P(A)\}$, and if A is a pointed locally boolean domain, then $P(A \multimap_s \Sigma) = \{\lfloor p \rfloor \mid p \in P(A) \wedge p \neq \bot\}$.*

PROOF: To show that all primes of $A \multimap \Sigma$ except $\bot$ have the form $\lfloor p \rfloor$ for some p, it suffices to show that every non-empty c.p. filter $X \in A \multimap \Sigma$ is a stable lub of primes of this form.

Each $X \in A \multimap \Sigma$ represents a stable map and thus has a trace — a subset $\mathrm{tr}(X) \subseteq X$ such that if $y \in X$, then there exists $x \in tr(X)$ such that $y \leq_s x$, and if $x \in \mathrm{tr}(X)$, $y \in X$ and $x \uparrow y$, then $x \leq_s y$. Note that if $x = \bigvee P(x) \in \mathrm{tr}(X)$, then there exists $p \in P(X)$ such that $p \in X$ and hence $x = p$ — i.e. every element of $\mathrm{tr}(X)$ is prime. Now it suffices to observe that if $p \in \mathrm{tr}(X)$, then $\lfloor p \rfloor \leq_s X$, and thus $\uparrow \{\lfloor p \rfloor \mid p \in \mathrm{tr}(X)\}$ and $X = \bigvee\{\lfloor p \rfloor \mid p \in \mathrm{tr}(X)\}$. If $p \in \mathrm{tr}(X)$, then $\lfloor p \rfloor \leq X$, so it remains to show $X \subseteq (\lfloor p \rfloor)^\top$. Suppose $x \in X$. Either $x \in \neg(\lfloor p \rfloor) \subseteq (\lfloor p \rfloor)^\top$ or $x \notin \neg(\lfloor p \rfloor)$, i.e. $x \notin \lceil \neg p \rceil$, and so $x \leq \neg p \leq p^\top$ and $p \leq \neg x \leq x^\top$. Thus $x \uparrow p$, and hence $p \leq_s x$, and so $x \in (\lfloor p \rfloor)^\top$ as required. $\square$

Proposition 3.23 *The categories $\mathcal{LBA}$ and $\mathcal{PLBA}_\mathcal{S}$ are dual. In other words, for each A, $\eta_A : A \rightarrow (A \multimap \Sigma) \multimap_s \Sigma$ is an isomorphism and for each pointed B, $\epsilon_B : B \rightarrow (B \multimap_s \Sigma) \multimap \Sigma$ is an isomorphism.*

PROOF: η_A restricts to a map from $P(A)$ to $P((A \multimap \Sigma) \multimap_s \Sigma)$: $\eta(p) = \{y \in A \multimap \Sigma \mid p \in y\} = \{y \in \Sigma^A \mid \lfloor p \rfloor \subseteq y\} = \lfloor \lfloor p \rfloor \rfloor$. By Lemma 3.22 this is a bijection.

Moreover, $p \uparrow q$ if and only if $\eta(p) \uparrow \eta(q)$: Suppose $p \leq q^\top$, then *either* $p \leq q$ — and hence $\lfloor\lfloor p \rfloor\rfloor \lfloor\lfloor q \rfloor\rfloor$ — *or* $p \leq \neg q$, and hence $p \notin \lceil \neg q \rceil$. Therefore $\lceil \neg q \rceil \notin \lfloor\lfloor p \rfloor\rfloor = \eta(p)$, and so $\lfloor\lfloor p \rfloor\rfloor \leq \lceil\lceil \neg q \rceil\rceil$, and $\eta(p) \leq \eta(q)^\top$ as required. The proof of the converse is similar.

Hence we can define the required inverses to the unit and co-unit: e.g. for $X \in (A \multimap \Sigma) \multimap_s \Sigma$, $\eta^{-1}(X) = \bigvee\{p \in P(A) \mid \eta(p) \leq_s X\}$. $\square$

We have already observed that lifting acts as a monad on $\mathcal{LBA}$ and as a co-monad on $\mathcal{PLBA}_\mathcal{S}$, and that the latter resolves as an adjunction between the inclusion of $\mathcal{PLBA}_\mathcal{S}$ into $\mathcal{LBA}$, and lifting from $\mathcal{LBA}$ into $\mathcal{PLBA}_\mathcal{S}$. The corresponding resolution of the lifting monad on $\mathcal{LBA}$ is as the adjunction of $(\_ \multimap \Sigma) : \mathcal{LBA} \to \mathcal{LBA}^{OP}$ with $(\_ \multimap \Sigma) : \mathcal{LBA}^{OP} \to \mathcal{LBA}$. (Since we have $A_\uparrow \cong (A_\uparrow \multimap_s \Sigma) \multimap \Sigma \cong (A \multimap \Sigma) \multimap \Sigma$.)

We can now derive the symmetric monoidal structure on $\mathcal{LBA}$ from the linear and affine function-spaces, using the duality with $\mathcal{PLBA}_\mathcal{S}$.

Definition 3.24 *We define $A \otimes B = (A \multimap (B \multimap \Sigma)) \multimap_s \Sigma$, and $f \otimes g = (f \multimap (g \multimap \mathtt{id}_\Sigma)) \multimap_s \mathtt{id}_\Sigma$ (which is bifunctorial because $\_ \multimap \_$ is a bifunctor).*

The unit of the monoid, I, is the one-point domain $*$ (which is terminal, so the model is affine). We have $A \otimes I = (A \multimap (I \multimap \Sigma)) \multimap_s \Sigma \cong (A \multimap \Sigma) \multimap_s \Sigma \cong A \cong I \otimes A$, $A \otimes B = (A \multimap B \multimap \Sigma) \multimap_s \Sigma \cong (B \multimap A \multimap \Sigma) \multimap_s \Sigma = B \otimes A$, and $A \otimes (B \otimes C) \cong (A \otimes B) \otimes C$.

Proposition 3.25 $(\mathcal{LBA}, I, \otimes, \multimap)$ *is a SMCC.*

PROOF: The adjunction $\_ \otimes A \dashv A \multimap \_$ arises from the natural isomorphism $\mathcal{LBA}(A \otimes B, C) = \mathcal{LBA}((A \multimap (B \multimap \Sigma)) \multimap_s \Sigma, C) \cong \mathcal{PLBA}_\mathcal{S}(C \multimap \Sigma, A \multimap (B \multimap \Sigma)) \cong \mathcal{LBA}(A, (C \multimap \Sigma) \multimap_s (B \multimap \Sigma)) \cong \mathcal{LBA}(A, B \multimap (C \multimap \Sigma) \multimap_s \Sigma) \cong \mathcal{LBA}(A, B \multimap C)$. $\square$

3.2 Modelling the ! of Linear Logic

Let $\mathcal{LBD}$ be the category of locally boolean domains and continuous and bistable functions. Clearly, every affine function is bistable, but not vice-versa and so $\mathcal{LBA}$ is a subcategory of $\mathcal{LBD}$. We shall now give a model of intuitionistic linear type theory by defining a (monoidal) right adjoint to this inclusion, and showing that $\mathcal{LBD}$ is cartesian closed. The intention is to define $!A = (A \Rightarrow \Sigma) \multimap_s \Sigma$, for which we need to know that if A is a locally boolean domain, then so is $A \Rightarrow \Sigma$ (we know it is a locally boolean order by Proposition 1.24).

Lemma 3.26 *For any family of pointed lbos, $\{A_i \mid i \in I\}$,*

$$(\Pi_{i \in I} A_i) \Rightarrow \Sigma \cong \Sigma_{k \in I}(A_i \Rightarrow_S \Pi_{i \in I - \{k\}} A_i \Rightarrow \Sigma)$$

(where $A \Rightarrow_S B$ denotes the lbo of strict and bistable functions from A to B.)

PROOF: Any monotone and bistable function into Σ is either constant or strict, and therefore by Lemma 3.10, k-strict for exactly one $k \in I$. $\square$

Lemma 3.27 *For any family of pointed lbos $\{A_{ij} \mid i \in I, j \in J_i\}$, there is an isomorphism*

$$\theta : \Pi_{i\in I}\Sigma_{j\in J_i}A_{ij} \Rightarrow \Sigma \cong \Sigma_{k\in I}\Pi_{l\in J_k}((A_{kl} \times \Pi_{i\in I-\{k\}}\Sigma_{j\in J_i}A_{ij}) \Rightarrow \Sigma)$$

such that if $f : \Pi_{i\in I}\Sigma_{j\in J_i}A_{ij} \to \Sigma$ is k-strict, then $\theta(f) = \mathtt{in}_k(\langle g_l \mid l \in j_k\rangle)$, where $g_l : (A_{kl} \times \Pi_{i\in I-\{k\}}\Sigma_{j\in J_i}A_{ij}) \to \Sigma$ is such that if $\pi_k(x) = \mathtt{in}_l(y_{kl})$, then $f(x) = g_l(\langle y_{kl}, \langle\pi_i(x) \mid i \in I - \{k\}\rangle\rangle)$.

Proof: We have an isomorphism obeying the above condition, obtained as follows:
$\Pi_{i\in I}\Sigma_{j\in J_i} \Rightarrow \Sigma \cong \Sigma_{k\in I}(\Sigma_{l\in J_k}A_{kl} \Rightarrow_s (\Pi_{i\in I-\{k\}}\Sigma_{j\in J_i}A_{ij} \Rightarrow \Sigma))$ (Lemma 3.26),
$\cong \Sigma_{k\in I}((\coprod_{l\in J_k} A_{kl}) \Rightarrow (\Pi_{i\in I-\{k\}}\Sigma_{j\in J_i}A_{ij} \Rightarrow \Sigma))$
$\cong \Sigma_{k\in I}\Pi_{l\in J_k}(A_{kl} \Rightarrow (\Pi_{i\in I-\{k\}}\Sigma_{j\in J_i}A_{ij} \Rightarrow \Sigma))$
$\cong \Sigma_{k\in I}\Pi_{l\in J_k}((A_{kl} \times \Pi_{i\in I-k}\Sigma_{j\in J_i}A_{ij}) \Rightarrow \Sigma)$. □

We can use this fact to prove that if A is a locally boolean domain then so is $A \Rightarrow \Sigma$ by showing that it can be reconstructed from sums and products.

Definition 3.28 *For a pointed lbd A, we define an ω-chain of locally boolean domains — $\{(A \Rightarrow \Sigma)_i \mid i \in \omega\}$ — approximating $A \Rightarrow \Sigma$, as follows:*
$(A \Rightarrow \Sigma)_0 = *$
$(A \Rightarrow \Sigma)_{n+1} = \Sigma_{q\in\mathsf{p}(A)}\Pi_{y\in\mathsf{m}(A_q^{\downarrow})}((A_{qy} \times \Pi_{p\in\mathsf{p}(A)-\{q\}}\Sigma_{x\in\mathsf{m}(A_p^{\downarrow})}A_{px}) \Rightarrow \Sigma)_n$

By Lemma 3.27, and Corollary 2.10, for all n, we have a retraction $\phi_n : (A \Rightarrow \Sigma)_n \trianglelefteq A \Rightarrow \Sigma : \psi_n$. Let $f^n = \psi_n(\phi_n(f))$ (so $f^n \leq f$).

Lemma 3.29 *Given a compact element x of pointed locally boolean domain A, for all continuous functions $f : A \to \Sigma$ such that $f(x) = \top$ there exists n such that $f^n(x) = \top$.*

Proof: We prove this by induction on $\#(x)$. Suppose $A \cong \Pi_{i\in I}\Sigma_{j\in J_i}A_{ij}$. If f is constant then $f^1 = f$ so by Lemma 3.10, we may assume f is k-strict for some $k \in I$. Then $\pi_k(x) \neq \bot$, and if $\pi_k(x) = \top$, $f^n(x) = \top$ for all $n \geq 1$. So suppose $\pi_k(x) = \mathtt{in}_l(y_{kl})$ for some $l \in J_k$ and $y_{kl} \in A_{kl}$. By Lemma 3.27, $\theta(f) = \mathtt{in}_k(\langle g_l \mid l \in J_k\rangle)$, where $g_l : (A_{kl} \times \Pi_{i\in I-\{k\}}\Sigma_{j\in J_i}A_{ij}) \to \Sigma$, and $f(x) = g_l(\langle y_{kl}, \langle\pi_i(x)| \, i \in I-\{k\}\rangle$. But $\#(\langle y_{kl}, \langle\pi_i(x) \mid i \in I-\{k\}\rangle\rangle) < \#(x)$ and so by inductive hypothesis, there exists n such that $g^n(\langle y_{kl}, \langle\pi_i(x) \mid i \in I - \{k\}\rangle = \top$, and so $f^{n+1}(x) = (g)^n(\langle y_{kl}, \langle\pi_i(x) \mid i \in I - \{k\}\rangle = \top$ as required. □

Proposition 3.30 *If A is a pointed locally boolean domain, then $A \Rightarrow \Sigma \cong \bigsqcup_{i\in\omega}(A \Rightarrow \Sigma)_i$.*

Proof: It suffices to show that $\bigvee_{n\in\omega} f^n = f$ — i.e. that if $f(x) = \top$ then $\bigvee_{n\in\omega} f^n(x) = \top$. But if $f(x) = \top$, then $f(y) = \top$ for some compact $y \leq_s x$, and by Lemma 3.29, $f^n(y) = \top$ for some n and so $\bigvee_{n\in\omega} f^n(x) \geq f^n(x) = \top$ as required. $\square$

Corollary 3.31 *If A is a locally boolean domain then $A \Rightarrow \Sigma$ is a locally boolean domain.*

Now we know that $\_ \Rightarrow \Sigma$ is a well-defined operation on locally boolean domains it is straightforward to show that it is in fact a functor with the following property.

Lemma 3.32 *$\_ \Rightarrow \Sigma : \mathcal{LBD} \rightarrow \mathcal{PLBA_S}^{OP}$ is left adjoint to $\_ \multimap_s \Sigma : \mathcal{PLBA_S}^{OP} \rightarrow \mathcal{LBD}$.*

By composing with the equivalence between $\mathcal{PLBA_S}^{OP}$ and $\mathcal{LBA}$ we have another adjunction; $! : \mathcal{LBA} \rightarrow \mathcal{LBD} = (\_ \Rightarrow \Sigma) \multimap_s \Sigma$ is left adjoint to $(\_ \multimap \Sigma) \multimap_s \Sigma$, or (by natural equivalence) the inclusion of $\mathcal{LBA}$ into $\mathcal{LB}$.

This is a symmetric monoidal functor, since $!A \otimes !B = (((A \Rightarrow \Sigma) \multimap_s \Sigma) \multimap ((B \Rightarrow \Sigma) \multimap_s \Sigma) \multimap \Sigma) \multimap_s \Sigma \cong (((A \Rightarrow \Sigma) \multimap_s \Sigma) \multimap B \Rightarrow \Sigma) \multimap_s \Sigma \cong (A \Rightarrow B \Rightarrow \Sigma) \multimap_s \Sigma \cong (A \times B \Rightarrow \Sigma) \multimap_s \Sigma = !(A \times B)$. Moreover, we have established that $\mathcal{LBD}$ is a cartesian closed (full) subcategory of $\mathcal{LB}$, since if A, B are locally boolean domains, then so is $A \Rightarrow B \cong !A \multimap B$. So this adjunction resolves a monoidal co-monad on $\mathcal{LBA}$, the co-Kleisli category of which is equivalent to $\mathcal{LBD}$.

4 Games as Locally Boolean Domains

We shall now define notions of game and strategies (basically AJ-games [1] or sequential data structures [6]) except that strategies may contain odd-length sequences, representing divergence or "going top") and show that a negation operation and extensional order can be defined on the set of strategies for a game, making it a locally boolean domain.

We shall use ε to refer to the empty sequence, $\sqsubseteq$ for the prefix ordering on sequences, and $s \sqsubseteq^E t$ to mean "s is an even-length prefix of t". A game

A is specified by a triple (M_A, λ_A, P_A), where M_A is a set of moves with a labelling function $\lambda_A : M_A \rightarrow \{P, O\}$ which partitions M_A into a disjoint sum of player moves $(M_A)^+$ and Opponent moves M_A^-. $P_A \subseteq M_A^{\circledast}$ is the set of plays of A, where $M_A^{\circledast}$ is the set of sequences over A which are finite, contain at most one occurrence of each move, alternating (i.e. P-moves precede O-moves and vice-versa), and contain at least as many Opponent as Player moves. Key examples are the "empty game" $\langle \varnothing, \varnothing, \{\varepsilon\}\rangle$, and the game with one (Opponent) move $o = \langle \{o\}, \langle o, O\rangle, \{\varepsilon, o\}\rangle$.

(Player) strategies on A are represented as sets of plays, using odd-length sequences to represent divergences.

Definition 4.1 *A Player strategy over a game A is a non-empty subset of P_A, subject to the conditions:*

- *Even-prefix-closure — if $s \sqsubseteq^E t \in \sigma$, then $s \in \sigma$.*
- *Even-branching: if $s, t \in \sigma$ and $s \sqcap t$ is odd-length then $s = t$. (Hence the only odd-length sequences in σ are of maximal length.)*

We shall write $\mathbf{strat}(A)$ *for the set of strategies over A.*

So, for instance, there are two strategies over o, $\{\varepsilon\}$ and $\{\varepsilon, o\}$.

Given a strategy σ, we shall write $E(\sigma)$ for its set of even-length sequences $E(\sigma)$ and $O(\sigma)$ for its set of odd-length sequences. Let $M(\sigma)$ be the set of odd-length sequences which extend a maximal sequence in $E(\sigma)$ — i.e. $M(\sigma) = \{sa \in P_A \mid s \in E(\sigma) \wedge \nexists t \in E(\sigma). sa \sqsubseteq t\}$. Then by definition, $O(\sigma) \subseteq M(\sigma)$, and if $S \subseteq M(\sigma)$ then $S \cup E(\sigma)$ is a well-defined strategy.

Definition 4.2 *For any strategy $\sigma : A$, its negation $\neg\sigma$ is defined: $E(\neg\sigma) = E(\sigma)$, $O(\neg(\sigma)) = M(\sigma) - O(\sigma)$.*

Clearly, negation is involutive, since $E(\neg\neg\sigma) = E(\neg\sigma) = E(\sigma)$, and $O(\neg\neg\sigma) = M(\sigma) - O(\neg\sigma) = M(\sigma) - (M(\sigma) - O(\sigma)) = O(\sigma)$. Of the two strategies in o, we have $\neg\{\varepsilon\} = \{\varepsilon, o\}$ and $\neg\{\varepsilon, o\} = \{\varepsilon\}$.

Definition 4.3 *We first define the extensional order on sequences:*
$s \leq t$ if s is even and $s \sqsubseteq t$, or t is odd and $t \sqsubseteq s$,
and then on strategies: $\sigma \leq \tau$ if $\forall s \in \sigma. \exists t \in \tau. s \leq t$

So in $\mathbf{strat}(o)$, for instance, we have $\{\varepsilon\} \leq \{\varepsilon, o\}$. More generally, for each game there is a $\leq$-least element $\bot$ — the empty strategy — and a $\leq$-greatest element $\top$, which contains every legal sequence consisting of at most one move.

Lemma 4.4 *If $\sigma \leq \tau$ then $\neg\tau \leq \neg\sigma$*

Proof: Suppose $\sigma \leq \tau$, and let $t \in \neg\tau$; we must show that there exists $s \in \neg\sigma$ such that $t \leq s$. Let s be the longest sequence in $\neg\sigma$ such that $s \sqsubseteq t$. If $s = t$, then we are done. If not, then s is of odd-length and hence $t \leq s$, since if s is of even-length, then it is either of maximal length in $E(\sigma)$ and hence there exists $sa \in O(\sigma)$ such that $s \leq t$, or there exists $sab \in E(\sigma)$ such that $sab \sqcap t = sa$. $\square$

Lemma 4.5 *For any game A, $\mathbf{Lbd}(A) = (\mathbf{strat}(A), \leq, \neg)$ is a pointed locally boolean order.*

Proof: For each σ, we define $\sigma^\top = E(\sigma) \cup M(\sigma)$, and thus $\sigma_\perp = E(\sigma)$, and if $\sigma \uparrow \tau$. It remains to show that if $\sigma \uparrow \tau$, then $\sigma \cup \tau$ and $\sigma \cap \tau$ are lubs and glbs with respect to $\leq$ (this includes the particular cases where $\tau = \neg\sigma$). In the first case, we need to show that $\sigma \cup \tau$ is even-branching, and thus a well-formed strategy. Suppose $s \in \sigma$ and $t \in \tau$ are maximal length sequences such that $s \sqcap t$ is odd-length. We must show that $s = t$. There exists $t' \in \tau^\top$ such that $s \leq t'$, and $s' \in \sigma^\top$ such that $t \leq s'$. If $s \sqsubseteq t'$, then $t' \sqcap t = s \sqcap t$ is odd length, and therefore $t' = t$, and so $s \sqcap t = s$, so s is also odd-length, and hence $s \leq t$ implies $s = t$. Similarly, if $t \sqsubseteq s'$, then $s = t$. If $t' \sqsubseteq s$ then t' is odd-length. If $t \sqsubseteq s \sqcap t$, then $t' \sqsubseteq t$ and so $t = t'$. If $t \not\sqsubseteq s \sqcap t$, then $t' \sqcap t = s \sqcap t$ and so $t = t'$ and $t \sqsubseteq s$. Similarly, $s' \sqsubseteq t$ implies $s \sqsubseteq t$ and so $s = t$ as required.

It is straightforward to see that $\sigma \cup \tau$ is a least upper bound, since $\sigma, \tau \leq \sigma \cup \tau$, and if $\sigma, \tau \leq \rho$, then for all $s \in \sigma \cup \tau$, either $s \in \sigma$ or $s \in \tau$ and we have $r \in \rho$ such that $s \leq r$.

Now consider $\sigma \cap \tau$. Clearly this is a strategy, but we need to argue that it is a greatest lower bound. $\sigma \cap \tau \leq \sigma, \tau$, since $\sigma \cap \tau \subseteq \sigma, \tau$. So suppose $\rho \leq \sigma, \tau$. Then if $r \in \rho$, there exist s, t such that $r \leq s, t$. If r is odd-length, then $s, t \sqsubseteq r$, and so either $s \sqsubseteq t$ or $t \sqsubseteq s$. But s, t are odd-length and $s, t \in \sigma \cup \tau$ and so $s = t \in \sigma \cap \tau$ as required.

If s, t are even length, then $r \sqsubseteq s, t$ and so $r \sqsubseteq s \sqcap t$. Since $s, t \in \sigma \cup \tau$, $s \sqcap t$ is even-length (otherwise $s \sqcap t = s = t$ but s, t are even-length). Hence $s \sqcap t \in \sigma \cap \tau$ and so $r \sqsubseteq s \sqcap t \in \sigma \cap \tau$ as required. $\square$

Corollary 4.6 *For any $\sigma, \tau : A$, $\sigma \leq_s \tau$ if and only if $\sigma \subseteq \tau$.*

Proof: If $\sigma \leq_s \tau$ then $\sigma \cup \tau = \tau$ and so $\sigma \subseteq \tau$. If $\sigma \subseteq \tau$ then by the definition, $\sigma \leq \tau$, and also $E(\sigma) \subseteq E(\tau)$ and so $\sigma_\perp \leq \tau_\perp$ as required. $\square$

The primes of $\mathbf{Lbd}(A)$ are the strategies consisting of a set of even prefixes of a single sequence. Thus the compact primes are the strategies which contain a unique maximal length sequence.

Definition 4.7 *Given $t \in P_A$, we shall write $\lfloor t \rfloor$ for the strategy $\{t\} \cup \{s \in P_A \mid s \sqsubseteq^E t\}$.*

Lemma 4.8 *σ is a compact prime in $\mathbf{Lbd}(A)$ if and only if there exists t such that $\sigma = \lfloor t \rfloor$.*

PROOF: For each compact strategy $\rho : A$, we have $\rho = \bigcup_{s \in \rho} \lfloor s \rfloor$, so if ρ is prime, we have $\rho = \lfloor s \rfloor$ for some s. Conversely, if s is any sequence then $\lfloor s \rfloor$ is compact prime, since if $\sigma \uparrow \tau$, and $\lfloor s \rfloor \subseteq \sigma \cup \tau$, then $s \in \tau$ or $s \in \sigma$. If $\sigma \downarrow \tau$, then suppose $\lfloor s \rfloor \subseteq \neg(\neg\sigma \cup \neg\tau)$. Then if s is even-length, $s \in E(\neg(\neg\sigma \cup \neg\tau)) = E(\sigma) \cup E(\tau)$. If s is odd-length, then $s \in M(\neg\sigma \cup \neg\tau) - O(\neg\sigma \cup \neg\tau)$ i.e. $s \in M(\neg\sigma)$ or $s \in M(\neg\tau)$, and $s \notin O(\neg\sigma)$ and $s \notin O(\neg\tau)$, and hence $s \in O(\sigma)$ or $s \in O(\tau)$ as required. $\square$

Proposition 4.9 *For any game A, $\mathbf{Lbd}(A)$ is a locally boolean domain.*

PROOF: This is straightforward, since for each strategy $\sigma : A$, we have $\sigma = \bigcup_{s \in \sigma} \lfloor s \rfloor$, and by definition of $\leq$, $\sigma \leq \tau$ if and only if for all $s \in \sigma$ there exists $t \in \tau$ such that $\lfloor s \rfloor \leq \lfloor t \rfloor$. Compact strategies are finite sets, and thus dominate finitely many elements. $\square$

4.1 Strategies as affine functions

We shall now give a definition of a category of games and strategies[2] along standard lines, and show that its morphisms correspond to affine functions on the associated locally boolean domains.

Definition 4.10 *From games A, B, we form a multiplicative product $A \otimes B$ as follows:*

- $M_{A \otimes B} = M_A + M_B$,
- $\lambda_{A \otimes B} = [\lambda_A, \lambda_B]$,

[2]This equivalent to a category of sequential data structures and *affine* sequential algorithms [6].

- $P_{A\otimes B} = \{t \in M_{A\otimes B}^{\circledast} \mid t\restriction A \in A \wedge t\restriction B \in B\}$.

The affine function-space $A \multimap B$ is formed as follows.

- $M_{A\multimap B} = M_A + M_B$,
- $\lambda_{A\multimap B} = [\overline{\lambda_A}, \lambda_B]$,
- $P_{A\multimap B} = \{t \in M_{A\multimap B}^{\circledast} \mid t\restriction A \in A \wedge t\restriction B \in B\}$.

For a family of games $\{A_i \mid i \in I\}$ we define the product $\Pi_{i\in I}A_i$, as follows:

- $M_{\Pi_{i\in I}A_i} = \coprod_{i\in I} M_{A_i}$,
- $\lambda_{\Pi_{i\in I}A_i} = [\lambda_{A_i} \mid i \in I]$,
- $P_{\Pi_{i\in I}A_i} = \coprod_{i\in I} P_{A_i}$

The empty product is the empty game, ε

Definition 4.11 *The category $\mathcal{G}$ has games as objects, and strategies on $A \multimap B$ as morphisms from A to B, with composition of $\sigma : A \to B$ and $\tau : B \to C$ defined:*

$$\sigma;\tau = \{s \in P_{A\multimap C} \mid \exists t \in (M_A + M_B + M_C)^*.t\restriction A, B \in \sigma \wedge t\restriction B, C \in \tau\}$$

The following result is standard.

Proposition 4.12 $(\mathcal{G}, \varepsilon, \otimes, \multimap)$ *is symmetric monoidal closed category with all small products.*

We shall now show that every strategy on $A \multimap B$ corresponds to an affine function from $\mathbf{Lbd}(A)$ to $\mathbf{Lbd}(B)$. For "points" in A — i.e. morphisms σ from ε to A, we shall just write $\sigma : A$.

Definition 4.13 *Given $\sigma : A \multimap B$, let $f_\sigma : \mathbf{Lbd}(A) \to \mathbf{Lbd}(B)$ be defined $f_\sigma(\tau) = \tau;\sigma$.*

We shall use the following lemma, which is in effect a version of the "switching condition".

Lemma 4.14 *Given $s \in P_{A\multimap B}$, s is even-length if and only if $s\restriction A$ and $s\restriction B$ are even-length, and s is odd-length if and only if exactly one of $s\restriction A$ and $s\restriction B$ is odd-length.*

Lemma 4.15 *For any $\sigma : A \multimap B$, f_σ is a monotone and affine function from* $\mathbf{Lbd}(A)$ *to* $\mathbf{Lbd}(B)$.

PROOF: For monotonicity, suppose $\rho : A \leq \tau : A$. Then given $r \in \rho;\sigma$, we have $s \in \sigma$ such that $s\restriction B = r$ and $s\restriction A \in \rho$. Hence there exists $t \in \tau$ such that $s\restriction A \leq t$. If $s\restriction A \sqsubseteq t$, then $s\restriction A$ is even-length and so $s\restriction A \in \tau$ and so $r = s\restriction B \in \tau$ as required. If $s\restriction A \not\sqsubseteq t$, t is odd-length and $t \sqsubseteq s\restriction A$. Hence there exists $s' \sqsubseteq^E s$ such that $s'\restriction A = t$, and $s'\restriction B \sqsubseteq s\restriction B = r$, and $s'\restriction B$ is odd-length, so $s\restriction B \leq s'\restriction B$ as required.

To establish that f_σ is also boundness-preserving, we show that for all $\tau : A$, $(f_\sigma(\tau))_\perp = (f_\sigma(\tau_\perp))_\perp$. Given $t \in (f_\sigma(\tau))_\perp = E(\tau;\sigma)$ there exists $s \in \sigma$ such that $s\restriction A \in \tau$ and $s\restriction B = t$. But since $s\restriction B$ is even-length, so are s and $s\restriction A$, and therefore $s\restriction A \in \tau_\perp$, and $s\restriction B \in (f_\sigma(\tau_\perp))_\perp$.

We have $f_\sigma(\tau) = \bigcup_{t\in\tau}\{u \in P_B \mid \exists s \in \sigma.s\restriction A = t \wedge s\restriction B = u\} = \bigvee\{f_\sigma(p) \mid p \in P(\tau)\}$. Thus f_σ preserves stable glbs and lubs. $\square$

The operation which takes $\sigma : A \multimap B$ to f_σ is functorial, since for all ρ, $(f_\sigma; f_\tau)(\rho) = f_\tau(f_\sigma(\rho)) = f_{\sigma;\tau}(\rho)$. Thus we have defined a functor $\mathbf{Lbd}$ from $\mathcal{G}$ to $\mathcal{PLBA}$, the category of pointed locally boolean domains and affine functions.

4.2 Locally Boolean Domains as Games

To show that the categories $\mathcal{G}$ and $\mathcal{PLBA}$ are in fact equivalent, we shall use the decomposition theorem to construct a game corresponding to each domain, and a strategy corresponding to each affine function. First, we give a definition of separated sum for games.

Definition 4.16 *The separated sum of a family of games $\{A_i \mid i \in I\}$ may be derived from the sum and product: $\Sigma_{i\in I}A_i = (\Pi_{i\in I}A_i \multimap o) \multimap o$.*

Plays in $\Sigma_{i\in I}A_i$ consist of a unique Opponent move, followed by a Player move corresponding to a choice of $i \in I$, followed by a play in A_i. Thus we may observe that strategies on $\Sigma_{i\in I}A_i$ are either $\top$, $\perp$, or a choice by Player of $i \in I$ together with a strategy on in A_i. ($\Sigma_{i\in I}A_i$ is a weak co-product on $\mathcal{G}$.)

Lemma 4.17 *For any family of games $\{A_i \mid i \in I\}$,* $\mathbf{Lbd}(\Pi_{i\in I}A_i) \cong \Pi_{i\in I}\mathbf{Lbd}(A_i)$, *and* $\mathbf{Lbd}(\Sigma_{i\in I}A_i) \cong \Sigma_{i\in I}\mathbf{Lbd}(A_i)$.

As studied in [15] we can take limits of sets of games which are directed with respect to an inclusion order.

Definition 4.18 *Let A, B be games. $A \trianglelefteq B$ if:*

- $M_A \subseteq M_B$,
- $\lambda_B = \lambda_A \restriction M_A$,
- $P_A = P_B \restriction M_A^{\circledast}$.

Then games ordered by $\trianglelefteq$ form a large cpo: Let Δ be a $\trianglelefteq$-directed set of games, then $\bigsqcup \Delta = (\bigcup_{A \in \Delta} M_A, \bigcup_{A \in \Delta} \lambda_A, \bigcup_{A \in \Delta} P_A)$

Lemma 4.19 *For any ω-chain of games $\{A_i \mid i \in I\}$, $\bigsqcup\{\mathbf{Lbd}(A_i) \mid i \in \omega\} \cong \mathbf{Lbd}(\bigsqcup_{i \in \omega} A_i)$.*

Definition 4.20 *Given a locally boolean domain D, we define the game of D, $\mathbf{Game}(D) = \bigsqcup_{i \in \omega} \mathbf{Game}_i(D)$, where:*
$\mathbf{Game}_0(D) = \{\varepsilon\}$, $\mathbf{Game}_{i+1}(D) = \Pi_{p \in \mathsf{p}(D)} \Sigma_{x \in \mathsf{m}(D_p^{\downarrow})} \mathbf{Game}_i(D_{px})$.

Proposition 4.21 *For all locally boolean domains then there is an isomorphism $\phi : D \cong \mathbf{Lbd}(\mathbf{Game}(D))$ and for all games A there is an isomorphism $\psi : A \cong \mathbf{Game}(\mathbf{Lbd}(A))$.*

Proof: is straightforward using Lemmas 4.17 and 4.19 and Corollary 2.10. □

Corollary 4.22 *Up to isomorphism in $\mathcal{G}$, every game can be generated using the product, sum and limit constructions.*

We shall now define **Game** as a functor, using the decomposition theorem.

Lemma 4.23 *Given $f : A \to B$, there exists a strategy $\sigma_f : \mathbf{Game}(A) \to \mathbf{Game}(B)$ such that $f_{\sigma_f} = \phi(f)$ and for all $\sigma : A \multimap B$, $\sigma_{f_\sigma} = \psi(\sigma)$.*

Proof: We first define σ_f for compact f by induction on $\#(f)$, using the decomposition theorem, following similar lines to the proof of Proposition 3.15.

First suppose that B is lifted — hence $\mathbf{Game}(B) = \Sigma_{x \in \mathsf{m}(B^{\downarrow})} B_x$.

- Suppose f is strict. Then if A is lifted, $\mathbf{Game}(A) = \Sigma_{x \in \mathsf{m}(A^{\downarrow})} A_x$, and since the lifted sum is a co-product in the category of lifted domains and strict maps, $f \cong [g_x \mid x \in \mathsf{m}(A^{\downarrow})]$ for some family $\{g_x : A_x \to B \mid x \in \mathsf{m}(A^{\downarrow})\}$, such that $\#(g_x) < g$ for all x and hence we may define $\sigma_f = [\sigma_{g_x} \mid x \in \mathsf{m}(A^{\downarrow})]$.

  If A is not lifted, then $\mathbf{Game}(A) = \Pi_{p \in \mathsf{p}(A)} A_p$, where each A_p is lifted, and by Lemma 3.11, $f \cong \pi_p; g$ for some $g : A_p \to B$. So let $\sigma_f = \pi_p; \sigma_g$.

- Suppose f is non-strict. Then either $f = \bot$ or $f = \top$ or by Lemma 3.7 $f \cong g; \mathtt{in}_x$ for some $x \in \mathsf{m}(B^{\downarrow})$ and $g : A \to B_x$ such that $\#(g) < \#(f)$ and so we may define $\sigma_f = \sigma; \mathtt{in}_x$.

Now if B is not lifted, $\mathbf{Game}(B) = \Pi_{p \in \mathsf{p}(B)} B_p$, where each B_p is lifted, and $\#(\pi_p(f)) \leq \#(f)$ for each p, so let $\sigma_f = \langle \sigma_{\pi_p(f)} \mid p \in \mathsf{p}(B) \rangle$.

Finally, for non-compact f we may define $\sigma_f = \bigvee \{\sigma_p \mid p \in P(A \multimap B) \wedge p \leq_s f\}$.

To show that $f_{\sigma_f} = \phi(f)$, we verify inductively over the definition of σ_f that for all $\tau : \mathbf{Lbd}(\mathbf{Game}(A))$, $f_{\sigma_f}(\tau) = \tau; \sigma_f = f(\phi^{-1}(\tau)) = \phi(f)(\tau)$.

We similarly verify that $\sigma_{f_\sigma} = \psi(\sigma)$ by induction over the definition of σ, using the fact that $\mathbf{Lbd}$ preserves product and weak co-product structure. □

So we may define $\mathbf{Game} : \mathcal{LBA} \to \mathcal{G}$ by $\mathbf{Game}(f) = \sigma$. Then $\mathbf{Game} \cdot \mathbf{Lbd} \cong \mathtt{id}_{\mathcal{G}}$ and $\mathbf{Lbd} \cdot \mathbf{Game} \cong \mathtt{id}_{\mathcal{LBA}}$ (and so $\mathbf{Game}$ is functorial). Thus we have shown the following.

Theorem 4.24 *The categories $\mathcal{PLBA}$ and $\mathcal{G}$ are equivalent.*

Corollary 4.25 *The category $\mathcal{G}$ is* isomorphic *to the category $\mathcal{G}^{ext}$ which has games as objects, and affine functions from $\mathbf{Lbd}(A)$ to $\mathbf{Lbd}(B)$ as morphisms from A to B.*

Remark 4.26 *Given $f : \mathbf{Lbd}(A) \to \mathbf{Lbd}(B)$, we may obtain a more direct characterization of a strategy σ such that $f_\sigma = f$ by considering the trace of f, which can be represented as subset of $P_A \times P_B$ such that $(s, t) \in \mathrm{tr}(f)$ if s is minimal in the prefix ordering such that $t \in f(\lfloor s \rfloor)$. This corresponds closely with Hyland and Schalk's notion of a strategy for an abstract game as a set of its reachable positions [8]: we may show that for any position $(s, t) \in \mathrm{tr}(f)$ there is a unique chain $(s_1, t_1), \ldots, (s_n, t_n) \in \mathrm{tr}(f)$ such that $s_1 \sqsubseteq s_2 \sqsubseteq \ldots \sqsubseteq s_n = s$ and $\bot \!-\!\!< t_1 \!-\!\!< t_2 \!-\!\!< \ldots \!-\!\!< t_n = t$, and from this we can reconstruct a unique sequence in $A \multimap B$.*

The equivalence also extends to the SMCC structure, as would be expected by uniqueness of adjoints, since we have shown that the two function-space constructions are equivalent.

Definition 4.27 *A strategy $\sigma : A \multimap B$ is* strict *if the first Player move in σ (if any) is always in A — i.e. if $ab \in \sigma$, then $b \in M_A$. The strict strategies form a subcategory $\mathcal{G}_S$ of $\mathcal{G}_S$. We also have a subcategory of $\mathcal{LG}_S$ of $\mathcal{G}$ in which the objects are "lifted games" — i.e. games with a unique opening move — and morphisms from A to B are strict strategies on $A \multimap B$.*

Lemma 4.28 *$f_\sigma : \mathbf{Lbd}(A) \to \mathbf{Lbd}(B)$ is strict if and only if $\sigma : A \multimap B$ is strict.*

Moreover, $\mathbf{Lbd}(A)$ is a lifted domain if and only if a is a lifted game. Thus the equivalence between $\mathcal{G}$ and $\mathcal{PLBA}$ cuts down to equivalences between $\mathcal{G}_s$ and $\mathcal{PLBA}_\mathcal{S}$ and $\mathcal{LG}_S$ and $\mathcal{LLBA}_\mathcal{S}$ (the category of lifted domains and strict functions).

Lemma 4.29 *The categories $\mathcal{G}$ and $\mathcal{LG}_S$ are dual.*

Proof: From $\mathcal{G}$ to $\mathcal{LG}_S^{OP}$ we have the functor $\_ \multimap o$. Since we have a natural isomorphism of hom-sets $G(A,B) \cong \mathcal{LG}_S(A \multimap o, B \multimap o)$, $\_ \multimap o$ has an inverse $\_ \multimap_s o : \mathcal{LG}_S^{OP} \to \mathcal{G}$ which removes the opening move from each game, and the opening two moves from each strategy. □

We note also that the duality between $\mathcal{LBA}$ and $\mathcal{PLBA}_\mathcal{S}$ cuts down to a duality between their subcategories $\mathcal{PLBA}$ and $\mathcal{LLBA}_\mathcal{S}$, and this duality factors through the equivalence with the categories of games (preserving the monoidal structure: for any games A, B, we have $A \otimes B \cong (A \multimap B \multimap o) \multimap_s o$).

We might also like to find a category of games which is equivalent to the whole category $\mathcal{LBA}$ of locally boolean domains and affine functions. An obvious candidate is the category $\mathsf{Fam}(\mathcal{G})$ — the coproduct completion of $\mathcal{G}$ — since we know that every locally boolean domain is isomorphic to a coproduct of pointed domains. But we can also use the duality between $\mathcal{LBA}$ and $\mathcal{PLBA}_\mathcal{S}$ to obtain an equivalent category of games more directly.

Corollary 4.30 *The categories $\mathcal{LBA}$ and $\mathcal{G}_s^{OP}$ are equivalent.*

4.3 Sequential Algorithms and Bistable Functions

We shall now outline the equivalence of the category of sequential algorithms and sequential data structures with the CCC of locally boolean domains. To define the former, we shall use a notion of monoidal co-monad ! due to Lamarche [11]; Curien [6] has shown that the co-Kleisli category obtained by the action of this ! on a category of games such as $\mathcal{G}$ is equivalent to a CCC of sequential algorithms defined in a more traditional fashion. So by showing the equivalence of the Lamarche ! on $\mathcal{G}$ with the ! obtained as the right adjoint of the inclusion of $\mathcal{LBA}$ into $\mathcal{LBD}$ we can show equivalence of bistable functions on locally boolean domains and sequential algorithms on sequential data structures.

Definition 4.31 *From a game A, we define a game $!A$ as follows:*

- $M_{!A} = P_A - \{\varepsilon\}$,
- $\lambda_{!A}(sa) = \lambda_A(a)$,
- $P_{!A} = \{s \in M_{!A}^{\circledast} \mid (tp \sqsubseteq s \wedge q \sqsubseteq p \wedge (q \neq \varepsilon) \implies \exists t'.t'q \sqsubseteq tp) \wedge (tp, t'q \sqsubseteq s \implies p \sqcap q \in P_A^E\}$.

Thus a play in $!A$ consists of a prefix-closed and even-branching set of plays of A, ordered as an alternating sequence. This suggests the definition of promotion for strategies on A (morphisms from I to A). For a play $s \in P_{!A}$, let $\mathbf{pmv}(s)$ be the set of Player moves (even-length plays) in s.

Definition 4.32 *Given $\sigma \in \mathbf{strat}(A)$, we define $\sigma^\dagger \in \mathbf{strat}(!A) = \{s \in P_{!A}^E \mid \mathbf{pmv}(s) \subseteq \sigma)\} \cup \{sa \in P_{!A}^O \mid a \cup \mathbf{pmv}(s) \subseteq \sigma\}$.*

Lemma 4.33 *For any strategies $\sigma, \tau \in \mathbf{strat}(A)$, if $\sigma \leq \tau$ then $\sigma^\dagger \leq \tau^\dagger$, and if $\sigma \updownarrow \tau$, then $\sigma^\dagger \updownarrow \tau^\dagger$ and $(\sigma \cup \tau)^\dagger = \sigma^\dagger \cup \tau^\dagger$ and $(\sigma \cap \tau)^\dagger = \sigma^\dagger \cup \tau^\dagger$.*

Proposition 4.34 $(!, (\_)^\dagger)$ *determines a unique monoidal co-monad on $\mathcal{G}$.*

Proof: As described in [6, 10], the promotion operation on points extends to general morphisms: for $\sigma : !A \to B$, we define $\sigma^\dagger : !A \to !B$, together with a natural transformation $\mathtt{der} : ! \to I$, such that $(!, \mathtt{der}, (\_)^\dagger)$ is a co-Kleisli triple. We also have a monoidal natural isomorphism between $!(A \times B)$ and $!A \otimes !B$. $\square$

Thus we have a cartesian closed co-Kleisli category $\mathcal{G}_!$ which is equivalent to a category of sequential algorithms on sequential data structures with a single error [6].

As in the linear case, for any strategy $\sigma :!A \rightarrow B$, we may define an associated function from $\mathbf{Lbd}(A)$ to $\mathbf{Lbd}(B)$ by composition. This is continuous and bistable [17, 7].

Lemma 4.35 *For any strategy $\sigma :!A \rightarrow B$, the map from $\mathbf{Lbd}(A)$ to $\mathbf{Lbd}(B)$ which takes$\tau \in \mathbf{strat}(A)$ to $\tau^\dagger;\sigma$ is monotone (continuous) and bistable.*

PROOF: This is immediate by Lemma 4.33, and the monotonicity and affineness of $f_\sigma : \mathbf{Lbd}(!A) \rightarrow \mathbf{Lbd}(B)$. $\square$

To define functors between $\mathcal{PLBA}_!$ and $\mathcal{G}_!$ we shall use the decomposition theorem to show that the action of the two bangs on prenex normal forms is equivalent.

Lemma 4.36 *For any family of pointed locally boolean domains, $\{A_{ij} \mid i \in I, j \in J_i\}$, there is an isomorphism $\alpha :!\Pi_{i\in I}\Sigma_{j\in J_i}A_{ij} \cong \Pi_{k\in I}\Sigma_{l\in J_k}!(A_{kl} \times \Pi_{i\in I-\{k\}}\Sigma_{j\in J_i}A_{ij})$ such that if $x \in \Pi_{i\in I}\Sigma_{j\in J_i}A_{ij} = \langle \mathtt{in}_{j_i}(x_i) \mid i \in I\rangle$, then $\alpha(x^\dagger) = \langle \mathtt{in}_{l_k}(x_k, \langle \pi_i(x) \mid i \in I - \{k\}\rangle) \mid k \in I\rangle$.*

PROOF: By Lemma 3.27, $\Pi_{i\in I}\Sigma_{j\in J_i}A_{ij} \Rightarrow \Sigma \cong \Sigma_{k\in I}\Pi_{l\in J_k}((A_{kl}\times\Pi_{i\in I-\{k\}}\Sigma_{j\in J_i}A_{ij}) \Rightarrow \Sigma)$, and so $!A = A \Rightarrow \Sigma \multimap_s \Sigma \cong \Pi_{k\in I}\Sigma_{l\in J_k}!(A_{kl} \times \Pi_{i\in I-\{k\}}\Sigma_{j\in J_i}A_{ij})$. $\square$

We also have an analogous fact for the ! on the category of games.

Lemma 4.37 *For any family of games $\{A_{ij} \mid i \in I, j \in J_i\}$ there is an isomorphism: $\beta :!\Pi_{i\in I}\Sigma_{j\in J_i}A_{ij} \cong \Pi_{k\in I}\Sigma_{l\in J_k}!(A_{kl} \times \Pi_{i\in I-\{k\}}\Sigma_{j\in J_i}A_{ij})$ such that if $\sigma : \Pi_{i\in I}\Sigma_{j\in J_i}A_{ij} = \langle \mathtt{in}_{j_i}(\sigma_i) \mid i \in I\rangle$, then $\beta(\sigma^\dagger) = \langle \mathtt{in}_{l_k}(\sigma_k, \langle \mathtt{in}_{j_i}(\sigma_i) \mid i \in I - \{k\}\rangle) \mid k \in I\rangle$.*

PROOF: (sketch)Plays in $!\Pi_{i\in I}\Sigma_{j\in J_i}A_{ij}$ start with an Opponent move $k \in I$ (corresponding to a play with a single move) and a Player move (a play of two moves) (kl), where $l \in J_k$. For each such $k \in I$ and $l \in J_k$, we have a map ϕ_{kl} from the moves of $!\Pi_{i\in I}\Sigma_{j\in J_i}A_{ij}$ (which are plays in $!\Pi_{i\in I}\Sigma_{j\in J_i}A_{ij}$), to plays in $A_{kl} \times \Pi_{i\in I-\{k\}}\Sigma_{j\in J_i}A_{ij}$, which removes the first two moves if they are kl, and is the identity, otherwise, i.e.
$\phi(s) = t$, if $s = kl \cdot t$,
$\phi(s) = s$, otherwise.

Thus we have an isomorphism of plays from $!\Pi_{i\in I}\Sigma_{j\in J_i}A_{ij}$ $\Pi_{k\in I}\Sigma_{l\in J_k}!(A_{kl}\times \Pi_{i\in I-\{k\}}\Sigma_{j\in J_i}A_{ij})$, where $\varepsilon \mapsto \varepsilon$, $(k) \mapsto (k)$, $(k)(kl) \cdot s \mapsto kl \cdot \phi_{kl}(s)$. $\square$

Proposition 4.38 *For any locally boolean domain A there is a natural isomorphism $\theta_A :!\mathbf{Game}(A) \cong \mathbf{Game}(!A)$.*

PROOF: By induction on normal form, using Lemma 4.37. $\square$

Hence also for any game A, we have a natural isomorphism $\gamma_A : \mathbf{strat}(!A) \cong\ !\mathbf{strat}(A)$.

Corollary 4.39 *The cartesian closed categories $\mathcal{G}_!$ and $\mathcal{PLBA}_!$ are equivalent.*

PROOF: We define functors $\mathbf{Game}_! : \mathcal{LBD}_! \to \mathcal{G}_!$ and $\mathcal{PLBA}_! : \mathcal{G}_! \to \mathcal{PLBA}_!$ as follows:
Given bistable $f :!A \to B$, we may define $\mathbf{Game}_!(f) :!\mathbf{Game}(A) \to \mathbf{Game}(B)$ to be $\theta_A; \sigma_f$, and for $\sigma :!A \multimap B$, we define $\mathbf{Lbd}_!(\sigma) = \gamma_A; f_\sigma$. Then $\mathbf{Lbd}_!$ is a functor, since for $\sigma :!A \multimap B$ and $\tau :!B \multimap C$, for all $\rho : A$ we have $\mathbf{Lbd}_!(\sigma^\dagger;\tau)(\rho^\dagger) = \rho^\dagger;(\sigma^\dagger;\tau) = \rho^\dagger;\sigma^\dagger;\tau = (\rho^\dagger;\sigma)^\dagger;\tau = (\mathbf{Lbd}_!(\sigma)^\dagger;\mathbf{Lbd}_!(\tau))(\rho^\dagger)$. (Because affine functions from $!A$ to B are bistable functions from A to B, we have $f = g$ if and only if $f(a^\dagger) = g(a^\dagger)$ for all $a \in A$.)

Since we have $\mathbf{Game}_! \cdot \mathbf{Lbd}_! \cong \mathtt{id}_{\mathcal{PLBA}_!}$ and $\mathbf{Lbd}_! \cdot \mathbf{Game}_! \cong \mathtt{id}_{G_!}$, and so this is an equivalence of (cartesian closed) categories, as required. $\square$

Corollary 4.40 *The cartesian closed categories of pointed locally boolean domains and bistable functions, and sequential algorithms on sequential data structures are equivalent.*

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